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FOUNDATIONS, MATHEMATICAL LOGIC

★ Copi, Irving M. *Symbolic logic*. The Macmillan Company, New York, 1954. xiii+355 pp. \$5.00.

The first half of this book (Ch. I-V) contains a detailed intuitive introduction to the logic of propositions, propositional functions and relations. Much attention is given to the translation from English into formal language and vice versa. Ch. VI-XI treat the propositional and first-order functional calculi up to Henkin's completeness proof for the latter. There are appendices on Boolean algebra and on the ramified theory of types.

A. Heyting (Amsterdam).

★ Lorenzen, Paul. *Einführung in die operative Logik und Mathematik*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXVIII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. vi+298 pp. DM 38.40.

This introduction to Logic and to the Foundations of Mathematics is based on a point of view which has been expounded previously by the author in a number of papers. While related to Intuitionism and to other constructivist trends, Lorenzen's approach does not coincide with any of these. He emphasizes above all the "operational" character of Mathematics, the fact that Mathematics depends, both in its foundations and at every step of its development, on the application of certain rules to finite systems, which may consist of numerals or of concrete objects of any other kind. The class of mathematical or metamathematical statements which the author regards as admissible (here called "definite") is rather wider than the class of decidable statements, which alone are regarded as legitimate by some constructivists.

The book first expounds the basic principles of Mathematical operations, including the construction of a sequence of numerals. The very method by which these numerals are obtained, rather than any specific axiom, is regarded as the basis for the principle of mathematical induction. The next chapter is concerned with a simple form of a calculus of natural deduction and this is expanded gradually into a calculus of propositions and a calculus of predicates. The negation of a statement X is by definition regarded as equivalent to the assertion that X entails a contradictory statement Λ (i.e. a statement which in turn implies all statements). Results which are obtained by the use of the tertium non datur are not rejected entirely but are granted a kind of second class status. (This point of view has lately been adopted also by some intuitionists.) Equality and descriptions are considered next and this is followed by a section on Modal Logic and on the Theory of Probability (for finite sample spaces). This completes the section on Pure Logic.

Arithmetic is introduced next, in the first instance as the theory of finite ordered sets ("systems"). The author remarks that certain transfinite ordinals also can be introduced within the methodical framework of the book,

thus renouncing a strictly finitary standpoint. Finite cardinals are then introduced (a cardinal is the "length" of a system) and it is shown that the operations with finite cardinals and ordinals are isomorphic. Rational numbers and negative numbers are defined as pairs, in the usual way. Real algebraic numbers are introduced by an original method which is based on the fact that any such number can be characterized, for some rational number a and some polynomial with integral coefficients $f(x)$, as the smallest root of $f(x)$ which is greater than a .

The author's theory of real numbers depends on the introduction of successive layers of classes (and of relations) of numbers. For a given layer of classes, the corresponding layer of real numbers is defined by means of Dedekind cuts. This procedure is related to the ramified theory of types, without axiom of reducibility. The transfinite cardinals lose their absolute meaning, any layer of real numbers turns out to be countable in some higher layer of relations. On this basis, the author then presents the elements of the theory of real functions, including Riemann and Lebesgue integration, and the theory of concrete real Hilbert space.

The last part of the book, entitled "General theory of structures" is concerned with the axiomatic approach. It follows from the author's basic tenets that a system of axioms can only be significant inasmuch as it refers to an actual situation within the edifice of Mathematics that was built up previously by a constructive (or "operational") method. This point of view is reflected also in the definition of a model. Various structures of Algebra and Topology are discussed in some detail.

The author has shown that much, though not all, of contemporary Mathematics can be "saved" by his methods. Whether or not the author's approach is too restrictive (as would appear to a classical mathematician) or too liberal (as would be held by some intuitionists) is inevitably a matter of opinion. It is a fact that there is no unanimity on these matters among those who reject the carefree approach of the classical mathematician. However, it is the great merit of the author that he has not been content with the statement of a particular philosophical attitude towards the foundations of Mathematics, but that he has followed up the practical consequences of his attitude in Logic and in Mathematics. Much of the value of the book lies precisely in the many detailed developments carried out by the author in support of his general thesis.

A. Robinson (Toronto, Ont.).

Lorenzen, Paul. *Zur Begründung der zweiwertigen Aussagenlogik*. Arch. Math. Logik Grundlagenforsch. 2 (1954), 29-32.

The author begins by expressing the view that the misunderstandings concerning the principle of the excluded middle arose from the consideration of mathematical statements. He gives a definition of descriptive statements and considers a propositional calculus which deals with such statements.

A. Rose (Nottingham).

Lorenzen, Paul. Zur Begründung der Modallogik. Arch. Math. Logik Grundlagenforsch. 2 (1954), 15-28.

The author considers definitions of modalities of the first and higher orders and derives properties of these modalities. He also uses similar methods to discuss reality, extension and intension. A. Rose (Nottingham).

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. VI^{tes}. Nederl. Akad. Wetensch. Proc. Ser. A. 57=Indagationes Math. 16 (1954), 389-396.

[For parts I-VI see same Proc. 55 (1952), 213-223, 459-467; 56 (1953), 1-11, 99-110, 378-388; 57 (1954), 2-8, 117-128; MR 14, 527, 1052; 15, 90; 16, 2, 438.] Ridder first shows that a necessary and sufficient condition for a structural inequality $p < q$ to be derivable in every Σ_j^* -structure is that the Σ_j^* -structures which do not contain more than $N_j(p, q)$ elements satisfy the relation $p < q$. $N_j(p, q)$ is a natural number which depends only on the type of structure considered and on the degree of complexity of p and q . He also proves the corresponding results for Σ_j^{*C} -structures ($j=2, 3, 4$; $k=1, 2$).

He then shows for the calculi $A^{*(s)}$, $A^{*(b)}$, $M^{*(s)}$, $M^{*(b)}$, $I^{*(s)}$, $I^{*(b)}$ that if $\{(M\mathcal{A}_1) \cdot (M\mathcal{A}_2) \cdot \dots (M\mathcal{A}_r)\}C\lambda$ is a theorem then, for some integer j ($1 \leq j \leq r$), $(M\mathcal{A}_j)C\lambda$ is a theorem. He then proves a similar theorem for the calculi A_1^* , M^* and I^* and shows that I^* is not reducible. Finally he considers the independence of some operations. Some dual results are given in the last paragraph. A. Rose.

Ridder, J. Die Gentzenschen Schlussverfahren in modalen Aussagenlogiken. I, II, III, III^{bis}. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 163-169, 170-177, 270-274, 275-276.

Ridder defines calculi $A_1^{(s)}$, $LG_1^{(s)}$, $A_2^{(s)}$, $LG_2^{(s)}$, $A_3^{(b)}$, $LG_3^{(b)}$, $M^{(s)}$, $LM^{(s)}$, $M^{(b)}$, $LM^{(b)}$, $I^{(s)}$, $LI^{(s)}$, $I^{(b)}$, $LI^{(b)}$, $K^{(s)}$, $LK^{(s)}$, $K^{(b)}$, $LK^{(b)}$ by adding certain modal postulates of previously defined calculi. The postulates are:

$$NXCX, N(XCY)C(NXCNY), \frac{\nu C\mathcal{B}}{\nu CN\mathcal{B}},$$

$$\frac{\mathcal{A} \cdot \mathcal{B} \rightarrow \mathcal{C}}{N\mathcal{A} \cdot \mathcal{B} \rightarrow \mathcal{C}}, \frac{N(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (N\mathcal{A} \rightarrow N\mathcal{B}), \frac{\nu \rightarrow \mathcal{B}}{\nu \rightarrow N\mathcal{B}}}{\nu \rightarrow \mathcal{B}}.$$

He shows that several pairs of calculi are equivalent and gives decision procedures following the Gentzen methods. Certain results concerning decision procedure for these calculi are applied to other propositional calculi. Finally it is shown that every decision method for the calculus \mathcal{S}_j or \mathcal{S}_j^* is also a decision method for the calculus \mathcal{S}_j , \mathcal{S}_j^* respectively ($j=2, \dots, 6$). A. Rose (Nottingham).

Quine, W. V. A proof procedure for quantification theory. J. Symb. Logic 20 (1955), 141-149.

This is essentially a new and very simple proof of Herbrand's theorem [Hilbert and Bernays, Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, pp. 157-158]. Let φ be a prenex formula of the first-order calculus. To φ there are associated certain formulas with free variables, called lexical instances of φ . Assuming that φ is false under every interpretation in every non-empty universe, the author constructs a truth-functionally inconsistent conjunction of lexical instances of φ . This construction is not finitary, because at one point it supposes a decision about the question whether, for a given system of truth-values, a given infinite system of formulas contains a false formula. A. Heyting (Amsterdam).

Łoś, J. On the extending of models. I. Fund. Math. 42 (1955), 38-54.

The author establishes some plausible results concerning those mathematical concepts which can be characterized by means of predicate logic of the first order and axioms in prenex form which do not contain existential quantifiers: such axioms are called 'open' by the author and 'universal' by Tarski [Nederl. Akad. Wetensch. Proc. Ser. A. 57 (1954), 572-581, 582-588; 58 (1955), 56-64; MR 16, 554]. The main results of the present paper were anticipated by Tarski, loc. cit. G. Kreisel.

Łoś, J., and Suszko, R. On the infinite sums of models. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 201-202.

This note is an addition to the paper by Łoś reviewed above. Let M_n ($n=1, 2, \dots$) be a sequence of models of an elementary theory T (axiom system formulated in the first-order predicate calculus), such that M_n is a submodel of M_{n+1} ; ΣM_n is the least model of which each M_n is a submodel. The authors state: a (closed) formula \mathcal{B} is either valid in ΣM_n or false in some M_n if and only if \mathcal{B} is a $\Pi\Sigma$ sentence, i.e. for some quantifier-free formula $A(a_1 \dots a_n, b_1 \dots b_m)$ in the notation of T , $\mathcal{B} \leftrightarrow (x_1) \dots (x_n) (Ey_1) \dots (Ey_m) A(x_1 \dots x_n, y_1 \dots y_m)$ is a tautology. This statement is not clear to the reviewer (unless 'tautology' means: theorem of T); for, if T is complete, every \mathcal{B} is either valid in all models or in none, yet \mathcal{B} need not be equivalent to a $\Pi\Sigma$ sentence in the pure predicate calculus (tautology). G. Kreisel (Princeton, N.J.).

Nelson, Raymond J. Simplest normal truth functions. J. Symb. Logic 20 (1955), 105-108.

Nelson develops methods for finding the simplest conjunctive and disjunctive normal forms of a formula. For the disjunctive case a method was previously developed by Quine [Amer. Math. Monthly 59 (1952), 521-531; MR 14, 440], but Nelson's method does not require the expansion of a formula into developed normal form as a preliminary step. A. Rose (Nottingham).

Yonemitsu, Naoto. Note on completeness of m -valued propositional calculi. Math. Japon. 3 (1954), 57-61.

It was shown by the reviewer [J. London Math. Soc. 26 (1951), 47-49; MR 12, 662] that if $m-1$ is prime then the degree of completeness of a formalisation of the m -valued Łukasiewicz propositional calculus satisfying certain conditions is equal to 3. It was later shown [Rose, ibid. 27 (1952), 92-102; MR 13, 811], unknown to author, that, for the general case, the degree of completeness is equal to $1+d(m-1)$. Yonemitsu proves some special cases of the last result. A. Rose (Nottingham).

Hu, Shih-Hua. Die endlichwertigen und functionell vollständigen Sub-systeme κ_n -wertigen Aussagenkalküls. Acta Math. Sinica 5 (1955), 173-191. (Chinese. German summary)

The author considers an infinite matrix R and two sequences P_n , Q_n ($n \geq 2$) of functionally complete finite-valued matrices, all with the designated value 0 ("true"). (1) R : for $m \geq k$, $C(m, k) = 0$, for $m < k$, $C(m, k) = k - m$; $F(m) = m + 1$. (2) P_n : $C^n(m, k) = C(m, k)$, $N^n(m) = n - 1 - m$, $T^n(m) = 1$, with $0 \leq m, k \leq n - 1$. (3) Q_n : $C^n(m, k)$, $F^n(m) = m + 1$ for $m < n - 1$, $F^n(n - 1) = n - 1$. It is proved that by substituting suitable combinations of F and C for C^n , N^n , T^n (resp. C^n , F^n), the proposition forms true in P_n (resp. Q_n) become coincident with those true forms in R .

which contain just such combinations of F and C . Well-known 2-valued and 3-valued systems are discussed to illustrate applications of the result. *Hao Wang.*

Moh, Shaw-Kwei. Some axiom systems for propositional calculus. *Acta Math. Sinica* 5 (1955), 117-135. (Chinese. English summary)

Five axiom systems of the propositional calculus are proposed such that by dropping (A and E), respectively weakening (B , C , D), one axiom, we get Heyting's calculus H , by dropping, respectively weakening, another axiom, we get Johanson's calculus J . System A : $C1$. $CpCqp$; $C2$. $CCpCqrCCpqCpr$; $C3$. $CCCpqpp$; $N1$. $CCpqCNqNp$; $N2$. $CpNNp$; $N3$. $CNpCpq$; deletion of $C3$ yields H ; deletion of $C3$ and $N3$ yields J . System E : same as A , with $C1$ and $C2$ replaced by $CCCpqrCsCCqCrtCqt$, $N1$ and $N2$ replaced by $N1^*$. $CCpNqCqNp$. System B : $C1$, $C2'$. $CCpqCCqrCpr$, $C3$, $N1'$. $CNpCpNNq$; $N2'$. $CCpNpNp$; $N3'$. $CNNpCNpp$; H is got by substituting $CCpCpqCpq$ for $C3$; J is got by substituting also $CNNpNp$ for $N3'$. System C : $C1''$. $CpCqCpr$; $C2''$. $CCsrCCCpqrCCpsr$; $N2'$, $N3$; H by substitution of $C2$ for $C2''$; J by substitution also of $CNpCpNq$ for $N3$. System D : C . $CCCpqrCCrCps$; N . $CCCpqCNpCCpNpr$; H by substitution of $CCCpqrCsCCqCrtCqt$ for C ; J by substitution of $N1^*$ for N .

Hao Wang (Cambridge, Mass.).

Moh, Shaw-Kwei. On the definition of primitive recursive functions. *Acta Math. Sinica* 5 (1955), 109-115. (Chinese. English summary)

The author obtains several variants of some results by R. M. Robinson [Bull. Amer. Math. Soc. 53 (1947), 925-942; MR 9, 221]: All primitive recursive functions can be defined from the successor function and the predecessor function ($P(0)=0$) (resp. the function $|u-x|$) by the composition schema and the following recursion schema: $\phi(u, 0)=u$ (resp. $\phi(u, 0)=0$), $\phi(u, x+1)=\psi(u, \phi(u, x))$. It is further observed that once we have addition and $|x-u|$, we can generally use a recursion schema with a first line $\phi(u, 0)=n_0$ (n_0 arbitrary constant given in advance) in place of $\phi(u, 0)=0$ and do as much.

Hao Wang (Cambridge, Mass.).

Kalmár, László. Über ein Problem, betreffend die Definition des Begriffes der allgemein-rekursiven Funktion. *Z. Math. Logik Grundlagen Math.* 1 (1955), 93-96.

The author displays a system of equations [in the sense of Kleene, Introduction to metamathematics, Van Nostrand, New York, 1952, pp. 264-265; MR 14, 525] containing a function symbol χ , such that the equations are true for all values of the number variables therein for one and only one assignment of values to $\chi(x)$, yet the function so determined (the unique solution of the system) is not recursive. *H. G. Rice* (Durham, N.H.).

Lacombe, Daniel. Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles. I, II, III. *C. R. Acad. Sci. Paris* 240 (1955), 2478-2480; 241 (1955), 13-14, 151-153.

The author extends the concept of recursiveness to functions of a real variable by defining such a function as an operator on characteristic functions of Dedekind classes of rationals, and as a mapping of nested sequences of intervals with rational endpoints. He announces several theorems, including the equivalence of these two definitions, and the continuity of the recursive functions

so defined. The purpose of some of the concepts introduced is not clear to the reviewer. *H. G. Rice.*

Mostowski, A. A formula with no recursively enumerable model. *Fund. Math.* 42 (1955), 125-140.

The author constructs a formula $F_1 \& F_2[X_0, Y_0]$ which is consistent over the predicate calculus, but has no recursively enumerable (r.e.) model. The non-logical constants of F_1 are $A(x)$, $B(x)$, $C(x)$, $F(x, y)$, and $G(x, y, z)$, F_2 contains in addition $D(x)$ and $E(x)$. There is a model of F_1 consisting of (finite) strings α, β, \dots of the letters a, b, c where $A(\alpha)$ means that α is a , $B(\beta)$ that β is b , $C(\gamma)$ that γ is c , $F(\alpha, \beta)$ that α and β are identical $G(\gamma, \alpha, \beta)$ that γ is $\alpha\beta$. The construction of F_2 is based on Post's representation [Amer. J. Math. 65 (1943), 197-215; MR 4, 209] of r.e. sets of integers by means of strings α, β, \dots where b and c function as markers and the integer n is represented by the string λ_n of n consecutive a . For r.e. disjoint sets X, Y , F_2 has a model in which $D(\lambda_n) \leftrightarrow n \in X$ and $E(\lambda_n) \leftrightarrow n \in Y$, and thus $n \in Y \rightarrow E(\lambda_n)$. The author obtains a long list of formal implications which ensure that decidable relations between strings of letters have counterparts in the axiomatic theory F_1 . If X_0 and Y_0 are r.e., but not separable by recursive sets [Kleene, Nederl. Akad. Wetensch., Proc. 53 (1950), 800-802; MR 12, 71] the author shows that $F_1 \& F_2(X_0, Y_0)$ does not possess a r.e. model. — The representation of decidable relations between strings of the letters a, b, c can be effected more simply by considering proposed r.e. arithmetic models of F_1 instead of the formal system F_1 : let $A_m(n_a)$, $B_m(n_b)$, $C_m(n_c)$ hold in the model, and let $(Eu)G_m(u; x, y, z)$ be the proposed model of $G(x, y, z)$; since $(x)(y)(Ez)G(z, x, y)$ is in F_1 , we have in the model $(x)(y)(Ez)(Eu)G_m(u; x, x, y)$ and therefore recursive functions ϕ and ψ satisfying $G_m[\psi(x, y); \phi(x, y), x, y]$. Thus, if the integers n and m represent the strings α and β , $\phi(n, m)$ represents $\alpha\beta$; in particular, there is a recursive λ such that the string λ_n is represented by $\lambda(n)$: $\lambda(1)=n_a$, $\lambda(n+1)=\phi(n_a, \lambda(n))$. Decidable relations between particular strings have numerical formulae as their counterparts in a r.e. model of F_1 . Now, the axioms of $F_2[X_0, Y_0]$ ensure that, if $D_m^* = \{n[\lambda(n) \in D_m]\}$ and $E_m^* = \{n[\lambda(n) \in E_m]\}$ then $D_m^* \supset X_0$ and $E_m^* \supset Y_0$, and D_m^*, E_m^* are mutually exclusive and exhaustive; if D_m and E_m are r.e., so are D_m^*, E_m^* and so they are recursive. This is excluded by the choice of X_0 and Y_0 .

Examples of other consistent formulae which do not possess r.e. models, were given previously by the author [Fund. Math. 40 (1953), 56-61; MR 15, 667] and in a slightly stronger version by the reviewer [Actes XIème Congrès Internat. Philos., 1953 v. XIV, North-Holland Publ. Co., Amsterdam, 1953, pp. 39-49; MR 15, 668], the former in September 1953, the latter in August. The author attributes priority to the reviewer: this seems unfounded since the time-lag is quite insignificant. *G. Kreisel.*

Markwald, W. Zur Eigenschaft primitiv-rekursiver Funktionen, unendlich viele Werte anzunehmen. *Fund. Math.* 42 (1955), 166-167.

Let $\varphi(k, x)$ be a general recursive function such that each primitive recursive function of one variable is $\varphi(k, x)$ for some k , and for fixed k , $\varphi(k, x)$ is primitive recursive. Mostowski showed [Bull. Acad. Polon. Sci. Cl. III. 1 (1953), 277-280, p. 280; MR 15, 667] that the set of all k such that $\varphi(k, x)$ has infinite range is a $\Sigma\Pi$ -set, but not a recursively enumerable set. In the present paper it is shown that this set is not a $\Pi\Sigma$ -set. *H. G. Rice.*

Ladner, G. On two problems of J. Schmidt. *Proc. Amer. Math. Soc.* 6 (1955), 647-650.

Verf. beantwortet durch Gegenbeispiele zwei Fragen aus einer Arbeit von J. Schmidt [Math. Nachr. 7 (1952), 165-182; MR 13, 904] negativ: 1. Es gibt erbliche Klassen von Mengen, die nicht als die Klasse aller K -unabhängigen Mengen einer Hüllenklasse K dargestellt werden können. 2. Es gibt nicht induktive Hüllenklassen K , für die die Klasse aller K -unabhängigen Mengen induktiv ist.

P. Lorenzen (Bonn).

Rasiowa, Helena. On a fragment of the implicative propositional calculus. *Studia Logica* 3 (1955), 208-226. (Polish. Russian and English summaries)

Consider the set of all tautologies with implication (C) as the only functor and such that after replacing C by the equivalence (E) they remain tautologous. This set is characterized by the product [in the sense of Kalicki, *J. Symb. Logic* 15 (1950), 174-181; MR 12, 663] of implication and equivalence matrices which is a four-valued matrix

C	1	2	3	4
*1	1	2	3	4
2	2	1	4	3
3	1	2	1	2
4	2	1	2	1

The formulae under study may be also described as those implicational tautologies in which each variable appears an even number of times. The author proves that every such tautology is a consequence (via modus ponens and substitution) of the following mutually independent axioms: 1. $CCpqCCqrCpr$; 2. $CqCpCpq$; 3. $CCCpqCCpqpp$. The addition of any implicational tautology independent from 1-3 (i.e., with a variable appearing an odd number of times) gives the complete implicational calculus. If one adds the standard rules for operating with quantifiers, one can deduce from 1-3 the complete implicational logic. The last result is attributed to Słupecki. H. Hiż.

Burks, Arthur W. Dispositional statements. *Philos. Sci.* 22 (1955), 175-193.

The author develops an analysis of causal dispositional statements. He introduces the idea of causal implication, pcq being defined by

pcq = it is causally necessary that $(p \supset q)$.

Noncausal necessity is defined by

Cp = it is causally necessary but not logically necessary that p .

Nonparadoxical causal implication is defined by

$pnpq = C(p \supset q) \cdot \sim C \sim p \cdot \sim Cq$.

He then analyses the causal dispositional statement, "This object is soluble in aqua regia". He applies the analysis to the uniformity-of-nature principle, to dispositional and extensional imperatives and to a theory concerning causality. The paper concludes with an analysis of probabilistic dispositional statements.

A. Rose (Nottingham).

Bing, Kurt. On arithmetical classes not closed under direct union. *Proc. Amer. Math. Soc.* 6 (1955), 836-846.

This paper contributes to the following unsolved problem: to characterize syntactically those first-order sentences G with the property D , that if G holds in each component of a direct union of algebras, then G holds for

the direct union. Horn [*J. Symb. Logic* 16 (1951), 14-21; MR 12, 662] proved that conditional sentences have the property D ; that is, prenex sentences with propositional matrix a conjunction of conditions $e_1 \& \dots \& e_n \rightarrow e_0$ (or, degenerately, $\sim(e_1 \& \dots \& e_n)$ or e_0), where each e_i is an elementary sentence (classically, an equation). Whether the converse holds is not known, although Horn showed this result best possible in terms of prenex form.

From a general lemma it is here deduced that, conversely, G with property D is equivalent to a conditional sentence, provided G is universal (prenex, all quantifiers universal), disjunctive (prenex, propositional matrix a disjunction of e_i and $\sim e_i$), or positive (prenex, matrix a conjunction of disjunctions of e_i).

The following sufficient condition for D is given. $G = \Omega F$, prenex form with Ω a string of quantifiers and $F = \bigwedge d_h, h \in H$. Here each d_h is a disjunction of e_i and $\sim e_i$, may be given the form $p_h \rightarrow q_h$, where $p_h = \bigwedge e_{h_i}, i \in P_h$ ($p_h = 1$ if $P_h = \emptyset$), and $q_h = \bigvee e_{h_j}, j \in Q_h$ ($q_h = 0$ if $Q_h = \emptyset$). Let $H = M \oplus N$, with $P_h \neq \emptyset$ for $h \in M$, and $r(h) \in Q_h \neq \emptyset$ for $h \in N$. Define $d_h' = \sim p_h$ for $h \in M$, and $d_h' = p_h \rightarrow q_{h(r(h))}$ for $h \in N$. (Thus each d_h' is a condition implying d_h .) Let $M = M_1 \oplus \dots \oplus M_s, F_h = \bigwedge d_h', h \in M_s + N$, and $G_s = \Omega F_s$. Then, if G implies all G_s , it follows that G has property D .

R. C. Lyndon (Ann Arbor, Mich.).

Mueller, Rolf. On the synthesis of a minimal representation of a logic function. Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass., Rep. AFRC-TR-55-104 (1955), iii+13 pp.

The author gives a method of determining a canonical form of a logic function such that no term contains a superfluous variable and the function is represented by the smallest number of such terms. The method is illustrated by means of two examples. A. Rose.

Gilmore, P. C., and Robinson, A. Metamathematical considerations on the relative irreducibility of polynomials. *Canad. J. Math.* 7 (1955), 483-489.

K sei ein Körper, der die folgende Bedingung erfüllt: Für jedes Polynom $p(t, x)$ in x mit Koeffizienten in $K(t)$ (es sei t transzendent über K), das keine Nullstellen in $K(t)$ hat, gibt es ein t^* , so dass $p(t^*, x)$ keine Nullstellen in K hat.

Verf. beweisen zunächst den metamathematischen Satz, dass es eine rein transzendente Erweiterung S' von $K(t)$ gibt, derart dass jede elementare Aussage, die in K gilt, auch in S' gilt. Als Anwendungen dieses Satzes ergeben sich dann leicht der Hilbertsche Irreduzibilitätssatz für K und zwei Verschärfungen hiervon für geordnete bzw. bewertete Körper. P. Lorenzen (Bonn).

Fraenkel, Abraham A. Integers and theory of numbers. Scripta Mathematica, New York, N.Y., 1955. vii+102 pp.

Contents: I. Natural numbers as cardinals. II. Natural numbers as ordinals. III. Theory of numbers. IV. Rational numbers.

As will be seen, this little book is mainly concerned with the logical foundations of the natural numbers and the rational numbers. The chapter on the theory of numbers is very elementary and limited in scope, but emphasizes those topics which will be of interest to a wide public. The exposition throughout the book is very clear. Two similar volumes are planned: one on the fundamental concepts of algebra and one on the fundamental concepts of the theory of sets. H. Davenport (London).

van Dantzig, D. Carnap's foundation of probability theory. *Synthese* 8 (1953), 459-470.

A discussion of Carnap's idea of probability, credibility or unique objective rational degree of belief, as described, for example, in his "Logical foundations of probability" [The Univ. of Chicago Press, 1950; MR 12, 664]. Probability, is intended to be defined in terms of a given object language, but Dantzig emphasises the difficulty of completing such a definition. He points out, among other things, that the claim of objectivity requires that probability₁ should be invariant under changes of object

language. [Cf. the reviewer's Probability and the weighing of evidence, Griffin, London, 1950, p. 48; MR 12, 837.]

The paper should be read in conjunction with some reference to R. Carnap [The continuum of inductive methods, Univ. of Chicago, 1952; MR 14, 4] and with W. Perks [J. Inst. Actuar. 73 (1947), 285-334, especially p. 308; this reference was apparently overlooked by Carnap; MR 9, 599].
I. J. Good (Cheltenham).

See also: Szmielew, p. 233.

ALGEBRA

Fujiwara, Tsuyoshi. Note on the isomorphism problem for free algebraic systems. *Proc. Japan Acad.* 31 (1955), 135-136.

An equational class A of abstract algebras consists of all algebras with certain operations in which certain relations hold identically. Every class A contains a free algebra $F_n(A)$ on n generators, for every cardinal n . A will be called trivial if $x=y$ holds identically in A . Theorem I. If A is not trivial, then $F_n(A) \cong F_m(A)$ for n infinite implies that $n=m$. Theorem II. If A contains a non-trivial class A of locally finite algebras, then $F_n(A) \cong F_m(A)$ implies $n=m$.
R. C. Lyndon (Ann Arbor, Mich.).

Combinatorial Analysis

Horner, Walter W. Addition-multiplication magic square of order 8. *Scripta Math.* 21 (1955), 23-27.

The author constructs a four-parameter family of latin squares of order eight such that in general if a_{ij} and b_{ij} are two such squares then a_{ij} and b_{ij} are orthogonal and the square $c_{ij} = a_{ij}b_{ij}$ has all its row sums and column sums equal. By suitable choice of the parameters and some rearranging of the rows and columns c_{ij} can be made to be additively and multiplicatively magic in the diagonals as well.
R. J. Walker (Ithaca, N.Y.).

Booth, A. D., and Booth, K. H. V. On magic squares. *Math. Gaz.* 39 (1955), 132-133.

It is proved that "the inverse of the matrix consisting of the elements of a magic square, is itself a magic square, whose row and column sums are the reciprocals of those of the original square". The definition of magic square used here is that the row sums and the column sums are all equal.
R. J. Walker (Ithaca, N.Y.).

Bose, R. C., and Clatworthy, W. H. Some classes of partially balanced designs. *Ann. Math. Statist.* 26 (1955), 212-232.

The paper deals with PBIB (partially balanced incomplete block) designs with two associate classes and $k > r \geq 2$, with $\lambda_1 = 1$ and $\lambda_2 = 0$. It is shown how the parameters for all such designs depend on the three integral parameters k , r , and t , which satisfy the restrictions that

- (i) $1 \leq t \leq r$,
- (ii) $rk(r-1)(k-1)/(t(k+r-t-1))$ is a positive integer.

For the particular case $r=3$ it is shown that all designs with $t=2$ or 3 necessarily exist, but if $t=1$, then the only possible value of $k > r$ is $k=5$. The designs with $t=2$ are well known lattice designs and those with $t=3$ are duals of certain balanced incomplete block designs. The design for $k=5$ is constructed, the only construction in the paper.

Other results are a less demanding definition of PBIB designs with two associate classes, and a lemma and five corollaries concerning the block structure of PBIB designs with two associate classes having $\lambda_1 = 1$ and $\lambda_2 = 0$, with $p_{11} = k-2$ (but with no special assumption about the relationship between r and k).
W. S. Connor.

Vartak, Manohar Narhar. On an application of Kronecker product of matrices to statistical designs. *Ann. Math. Statist.* 26 (1955), 420-438.

The Kronecker product $A \times B$ of two matrices $A = (a_{ij})$, $B = (b_{ij})$ is the matrix $(a_{ij}b_{ij})$. The Kronecker product of the incidence matrices of two designs can be considered as the incidence matrix of another design. This design is called the product of the other two designs. The products of various designs are considered in the paper. Let $N_1 = (1, 1, \dots, 1)$ and $N_2 = N_1'$ the transpose of N_1 . The authors main results are. If N_2 is a PBIB with parameters $v, b, r, k, \lambda_1, \lambda_2, p_{11}$ with s associate classes then $N = N_2 \times N_2$ is a PBIB with at most $s+1$ associate classes. The design N has exactly $s+1$ associate classes if the λ_i are all distinct and less than r . The parameters of N are explicitly given in the paper. In particular, if N_2 is a BIB then N is a singular group divisible design. The product of two PBIB with s and t associate classes is a PBIB with at most $s+t+st$ associate classes. The author also gives conditions for this design to have exactly $s+t+st$ associate classes and gives the parameters of the product explicitly in terms of those of the factors. Several examples are given of the construction of designs by means of the Kronecker product.
H. B. Mann (Columbus, Ohio).

Seiden, Esther. On the maximum number of constraints of an orthogonal array. *Ann. Math. Statist.* 26 (1955), 132-135; correction 27 (1956), 204.

R. C. Bose and K. A. Bush [same Ann. 23 (1952), 508-524; MR 14, 442] showed how to make use of the maximum number of points, no three of which are collinear, in finite projective spaces for the construction of orthogonal arrays. This enabled them to construct an orthogonal array (81, 10, 3, 3). They proved, in this case, that the maximum number of constraints, by any method of construction, does not exceed 12. In the present paper it is shown that an orthogonal array (81, 10, 3, 3), constructed by the geometrical method, cannot be extended to an 11-rowed orthogonal array. It is then undertaken to prove that the number of constraints k in an orthogonal array (81, k , 3, 3), constructed by any method, cannot exceed 11. However, the author has indicated by private communication that there is an error in the proof, but that the error does not nullify the truth of the assertion.
W. S. Connor (New Brunswick, N.J.)

See also: Hughes, p. 234.

Linear Algebra, Polynomials, Invariants

San Juan Llosá, Ricardo. The "simplex" method of linear programming. *Rev. Ci. Apl.* 8 (1954), 481-492; corrections 9 (1955), 133-136. (Spanish)

Criterion, using signs of minors up to order h , for a member of a base not to reoccur in the next h bases. (The corrections consist of a rewritten section and of a list of 81 errors; however, the new version, and the bibliography, contain further errors.) *T. S. Motzkin.*

Berge, Claude. Sur une propriété des matrices doublement stochastiques. *C. R. Acad. Sci. Paris* 241 (1955), 269-271.

Using a result of Ostrowski [*J. Math. Pures Appl.* (9) 31 (1952), 253-292; MR 14, 625] it is shown that every doubly stochastic matrix $A=(a_{ik})$, $a_{ik} \geq 0$, $\sum_i a_{ik} = \sum_k a_{ik} = 1$, is the product of finitely many "elementary" ones, i.e. in which at most four elements are $\neq 0, 1$. Reviewer's remark: equivalently, A is, for every $\delta > 0$, a product of "almost-transpositions" $(1-\varepsilon)T + \varepsilon I$, $\varepsilon < \delta$ and "half-transpositions" ($\varepsilon = \frac{1}{2}$), where T is obtained from the identity I by interchanging two rows. *T. S. Motzkin.*

Ostrowski, Alexandre. Sur les déterminants à diagonale dominante. *Bull. Soc. Math. Belg.* 1954, 46-51 (1955).

This is a lecture reporting on several older and several recent theorems concerning determinants with dominant main diagonal. *O. Taussky-Todd* (Washington, D.C.).

Ostrowski, Alexander. Über Normen von Matrizen. *Math. Z.* 63 (1955), 2-18.

Continuing earlier work by the author [*C. R. Acad. Sci. Paris* 232 (1951), 786-788; MR 12, 596] and W. Gautschi [*Duke Math. J.* 20 (1953), 127-140, 375-379; *Compositio Math.* 12 (1954), 1-16; MR 15, 94; 16, 105, 326] the definitions for the norm and bound of a square matrix A are investigated axiomatically. A "general" norm $N(A)$ satisfies $N(A) > 0$ for $A \neq 0$, $N(cA) = |c|N(A)$ for scalars c and $N(A+B) \leq N(A) + N(B)$. A "multiplicative" norm $M(A)$ further satisfies $M(AB) \leq M(A)M(B)$. It is shown that for multiplicative norms the axiom $M(A) > 0$ can be replaced by $M(A) \geq 0$ and at least one A_0 with $M(A_0) > 0$. Next it is shown that a multiplicative norm is never exceeded by the absolute value of a characteristic root, a result also obtained by Faddejewa [*Computational methods of linear algebra*, Gostehizdat, Moscow-Leningrad, 1950; MR 13, 872] and in special cases, even for non-multiplicative norms, by Gautschi. Analogous definitions hold for the norm $h(\xi)$ of a vector ξ . A special vector norm is obtained from a matrix norm by considering the square matrix whose first row (or column) is the given vector and the remainder zeros. On the other hand, from any definition of $h(\xi)$ a matrix norm, the so called "bound norm" can be obtained by taking $\sup_{\xi \neq 0} h(A\xi)/h(\xi)$. It is shown that no $M(A)$ is exceeded by the bound norm formed from the vector norm induced by $M(A)$. Even more generally an "unsymmetric" bound norm $\sup_{\xi \neq 0} h(A\xi)/h_1(\xi)$ can be obtained from two vector norms h, h_1 . For general norms $N(A)$ an estimate of $N(A^p)$ is carried out for the purpose of proving that $[N(A^p)]^{1/p}$ tends to the absolute value of the dominant characteristic root of A . The three frequently used norms $\sum_{i,j} |a_{ij}|$, $(\sum_{i,j} |a_{ij}|^2)^{1/2}$, $\max |a_{ij}|$ are considered as special cases of the Hölder matrix norm $|A|_p = (\sum |a_{ij}|^p)^{1/p}$ ($p \geq 1$). This norm is now studied even for rectangular matrices. It is

shown that $|AB|_p \leq |A|_p |B|_p$ for $p \leq 2$, while for $p > 2$ this inequality ceases to hold in general unless A is simply a row and B a column. For $p \geq 2$, $1/p + 1/q = 1$, we have $|AB|_p \leq |A|_p |B|_q$ and $|AB|_q \leq |A|_q |B|_p$. For $p < 2$ the first of these inequalities ceases to hold in general unless B is a column, the second unless A is a row. These theorems are generalized to products of more factors, the case of an even number of factors differing from the case of an odd number. The paper contains further results concerning the norm $[\sum_i (\sum_j |a_{ij}|^q)^{1/q}]^{1/p}$, $1/p + 1/q = 1$, and results concerning the norms derived from Hölder vector norms. A connection between the results discussed and theorems concerning linear transformations in Hilbert space is indicated. [Another recent axiomatic study of norms was made by A. S. Housholder, Oak Ridge Nat. Laboratory Rep. ORNL 1756 (1954); MR 16, 211.]

O. Taussky-Todd (Washington, D.C.).

Goddard, L. S. Note on a matrix theorem of A. Brauer and its extension. *Canad. J. Math.* 7 (1955), 527-530.

Let A be an $n \times n$ matrix, and let both K and X be $n \times r$ matrices, where the columns of X are characteristic vectors of A , all belonging to one and the same characteristic root λ . Write $B = XK'$, and let μ_1, \dots, μ_σ be the non-zero roots of B . The author shows, if $f(A, B)$ is any scalar polynomial in A and B , that the roots of $f(A, B)$ are $f(\lambda, \mu_i)$ ($i=1, \dots, \sigma$), and further $f(\lambda, 0)$, $f(\lambda_1, 0)$, \dots , $f(\lambda_r, 0)$ with multiplicities $\tau - \sigma$, m_1, \dots, m_r , respectively (here τ, m_1, \dots, m_r denote the multiplicities of the roots $\lambda, \lambda_1, \dots, \lambda_r$ of A , respectively). [The theorem can also be stated in the following form: A and B can be simultaneously transformed (similarity transformation) into triangular form in such a way that B gets zeros at all diagonal places where A has eigenvalues $\neq \lambda$.] The special case $r=1$, $f(A, B) = A+B$ is a theorem of A. Brauer [*Duke Math. J.* 19 (1952) 75-91; MR 13, 813]. A second theorem deals with $A+XK'$ in the case that the columns of X are associated with distinct characteristic roots of A .

N. G. de Bruijn (Amsterdam).

Kotelyanskii, D. M. On the disposition of the points of a matrix spectrum. *Ukrain. Mat. Ž.* 7 (1955), 131-133. (Russian)

Let $A=(a_{ij})$ be an $n \times n$ matrix of complex elements; $A \neq 0$; let $b_{ij}=|a_{ij}|$; let $B=(b_{ij})$; let μ be the dominant characteristic root of B ($\mu > 0$). Then every characteristic root of A lies in one of the n circles $C_i: |z - a_{ii}| \leq |\mu - a_{ii}|$. This theorem does not include, nor is it included in, the known theorems of similar nature. Of course, it is more complicated to apply than e.g. Gersgorin's theorem [*Izv. Akad. Nauk. Otd. Mat. Estest. Nauk* (7) 1931, 749-754].

J. L. Brenner (Pullman, Wash.).

Hukuhara, Masuo. Sur les valeurs propres des endomorphismes de l'espace vectoriel. *Proc. Japan Acad.* 31 (1955), 126-127.

For a linear transformation T , let $N(T, m, \lambda)$ be the null space of $(T - \lambda I)^m$. Let f be a polynomial and let the roots of $f(x) = \lambda$ be λ_i with multiplicities n_i . Then $N(f(T), m, \lambda) = \sum N(T, mn_i, \lambda_i)$. In the proof the author refers to an earlier paper [*J. Fac. Sci. Univ. Tokyo. Sect. I.* 7 (1954), 129-192; MR 16, 992]. *I. Kaplansky.*

Morinaga, Kakutaro, and Nôno, Takayuki. On the non-commutative solutions of the exponential equation $e^x e^y = e^{x+y}$. II. *J. Sci. Hiroshima Univ. Ser. A.* 18 (1954), 137-178.

Let x, y be matrices such that $xy \neq yx$, $e^x e^y = e^{x+y}$.

1) The authors show how to obtain (inductively) all triangular $n \times n$ matrices which satisfy this relation. Triangular solutions have previously been given by Kakar [Rend. Circ. Mat. Palermo (2) 2 (1954), 331-345; MR 16, 4.] Certain properties of these triangular solutions are also given. For example, if λ_i are the characteristic roots of x and μ_i are the characteristic roots of y , then

$$(\mu_i - \mu_j)[\exp(\lambda_i - \lambda_j) - 1] = (\lambda_i - \lambda_j)[\exp(\mu_i - \mu_j) - 1];$$

the total set of properties given is quite extensive. 2) All solutions of $e^x e^y = e^{x+y} = e^y e^x$ are found if x, y are 3×3 matrices of complex numbers. There are other solutions besides $\text{diag}(x', 1)$, $\text{diag}(y', 1)$ with $x', y', 2 \times 2$ matrices. 3) All solutions are found for an algebra over complex numbers, such that $x^3 = \alpha^2 x$, $y^3 = \beta^2 y$, $x^2 y = y x^2 = \alpha^2 y$, $xy^2 = y^2 x = \beta^2 x$; this generalizes results in part I of the same paper [same J. 17 (1954), 345-358; MR 16, 558].

J. L. Brenner (Pullman, Wash.).

Morinaga, Kakutaro, and Nôno, Takayuki. On the complex orthogonal transformations. J. Sci. Hiroshima Univ. Ser. A. 18 (1955), 349-377.

Let M be a complex orthogonal matrix. The authors represent M as the exponential of a matrix K whose blocks (in the Jordan canonical form) have certain special properties, which it would be too long to describe here. Using this, they obtain a new proof of the fact that M belongs to a one-parameter group of orthogonal matrices if and only if it is the square of an orthogonal matrix (or, which amounts to the same, of a rotation). Moreover, assuming that M is a rotation, they obtain estimates of the number of plane rotations in which it is possible to decompose M in terms of the properties of the matrix K .

C. Chevalley (Paris).

Ionescu, D. V. Une identité importante et la décomposition d'une forme bilinéaire en une somme de produits. Gaz. Mat. Fiz. Ser. A. 7 (1955), 303-312. (Romanian. Russian and French summaries)

Let the function $\phi(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n)$ be homogeneous of degree p in the x 's and of degree q in the y 's. Set $\phi_i = \partial\phi/\partial x_i$, $\phi_j = \partial\phi/\partial y_j$, $\lambda_{ij} = \partial^2\phi/\partial x_i \partial y_j$; let $\Delta = \Delta_n$ stand for the determinant of the λ_{ij} 's, while Δ_r stands for the determinant of the first r rows and columns of Δ , with $\Delta_0 = 1$. Finally, let D_r stand for the determinant obtained from Δ_r by replacing in the last column the λ_{ij} 's by ϕ_i 's, while D_r^* is similarly obtained from Δ_r by replacing the λ_{ij} 's of the last row by ϕ_j 's. Then (*) $\phi = \sum_{i=1}^n D_i D_i^* / \Delta_{i-1} \Delta_i$ holds in any domain $D(a_i \leq x_i \leq b_i; a'_i \leq y_i \leq b'_i)$, where all λ_{ij} exist and are continuous and where $\Delta_r \neq 0$ for all $r \leq n$. If at every point of a certain subdomain $D''CD$ the rank of the matrix of the λ_{ij} is $r < n$, then only the first r terms remain in (*). In particular, if ϕ is the bilinear form $\sum_{i,k=1}^n a_{ik} x_i y_k$, then $D_i = P_i(x)$, $D_i^* = Q_i(y)$, where P_i and Q_i are linear. Hence, (*) becomes

$$\sum_{i,k=1}^n a_{ik} x_i y_k = \sum_{i=1}^r P_i(x) Q_i(y) / \Delta_{i-1} \Delta_i$$

with $r \leq n$. In case $a_{ik} = a_{ki}$, then $P_i = Q_i$, and, if furthermore $x_i = y_i$, then (*) reduces to the classical decomposition of a quadratic form into a sum of squares. In the case of a hermitian form $\sum_{i,k=1}^n a_{ik} \bar{x}_i y_k$, with $a_{ik} = \bar{a}_{ki}$, one obtains $P_i(x) = \bar{Q}_i(y)$, hence

$$\sum_{i,k=1}^n a_{ik} \bar{x}_i y_k = \sum_{i=1}^r \bar{Q}_i(x) Q_i(y) / \Delta_{i-1} \Delta_i$$

and

$$\sum_{i,k=1}^n a_{ik} \bar{x}_i x_k = \sum_{i=1}^r |Q_i(x)|^2 / \Delta_{i-1} \Delta_i$$

with real Δ_i 's. For the proof of (*) one uses the generalized Euler identities $\sum_{j=1}^n x_j \phi_j / p = \sum_{j=1}^n y_j \psi_j / q = \phi$, whence follows

$$\phi = -\frac{1}{\Delta_n} \begin{vmatrix} \lambda_{11} & \dots & \lambda_{1n} & \phi_1 \\ \vdots & & \vdots & \vdots \\ \lambda_{n1} & \dots & \lambda_{nn} & \phi_n \\ \psi_1 & \dots & \psi_n & 0 \end{vmatrix}.$$

Reducing the submatrix of the λ_{ij} to triangular form and expanding the determinant, one obtains (*).

E. Grosswald (Philadelphia, Pa.).

Bendrikov, G. A., and Teodorčik, K. F. Laws of migration of roots of linear algebraic equations of third and fourth degree for continuous variation of the free term. Avtomat. i Telemekh. 16 (1955), 288-292 (2 plates). (Russian)

The authors discuss the dependence of the roots of third and fourth degree polynomials upon the constant term.

R. Bellman (Santa Monica, Calif.).

Howald, Mario. Die akzessorische Irrationalität der Gleichung fünften Grades. Comment. Math. Helv. 29 (1955), 279-297.

Felix Klein's theory of the quintic

$$G(x) = \sum_{r=0}^5 a_r x^{5-r} = \prod_{i=1}^5 (x - \alpha_i) \quad (a_0 = 1)$$

includes as one of its features the construction of a certain rational function $d = d(\alpha_1, \dots, \alpha_5)$ over the field of rational functions of $\alpha_1, \dots, \alpha_5$, having the property that it remains unaltered when the α 's undergo an even permutation. The function d , whose square root is the accessory irrationality referred to in the title, depends upon an arbitrary rational function $w(\alpha_1, \dots, \alpha_5)$ which is invariant under cyclic permutations of the α 's. The paper is concerned with the explicit determination of d , when w is given. The details are very complicated (d contains in general 1830 terms) and are mastered only by skilfully exploiting the structure of the icosahedral group.

W. Ledermann (Manchester).

Perron, Oskar. Neuer Beweis zweier Sätze von Zariski über die Multiplizität einer Lösung von k Gleichungen mit k Unbekannten. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1954, 179-199 (1955).

The author gives a new proof of the following theorem, due to Zariski [Trans. Amer. Math. Soc. 41 (1937), 249-265]: let F_1, \dots, F_n be polynomials in n variables X_1, \dots, X_n ; assume that the hypersurfaces $F_1 = 0, \dots, F_n = 0$ have only a finite number of common points and that the terms of lowest degree in F_i are of degree $r_i > 0$; let f_i be the homogeneous component of degree r_i of F_i ; then the origin is of multiplicity $\geq r_1 \dots r_n$ in the intersection of $F_1 = 0, \dots, F_n = 0$, and the lower bound is reached if and only if the origin is the only common point to the hypersurfaces $f_1 = 0, \dots, f_n = 0$. The proof (like Zariski's original one) is computational; the result can be established in a few lines by using the theory of specialisations.

C. Chevalley (Paris).

See also: Dieudonné, p. 236; Arrighi, p. 252; Bellman, p. 274.

Lattices

Funayama, Nenosuke. Notes on lattice theory. III. "Modular" or "distributive" lattice-homomorphisms. Bull. Yamagata Univ. (Nat. Sci.) 1 (1951), 219-222. (Japanese summary)

[For parts I and II see MR 17, 286.] The author calls a lattice homomorphism modular or distributive if the image lattice is modular or distributive. He shows that a lattice homomorphism is modular or distributive if and only if it annuls all non-modular quotients or all non-distributive quotients. A quotient a/b is said to be non-modular if there exists a lattice element c such that $a \cup c = b \cup c$ and $a \cap c = b \cap c$. The same quotient is called non-distributive if there exist lattice elements x, y , and z such that $x \cup y = y \cup z = z \cup x = a$, and $x \cap y = y \cap z = z \cap x = b$. He thus determines the "smallest" modular and distributive homomorphisms of a lattice. As a corollary he obtains a converse of a theorem of G. Birkhoff and the reviewer to the effect that the correspondence which assigns to each lattice element x the set of all prime dual ideals of an arbitrary collection K of such ideals which contain x , is both a meet-representation and a join-representation of the lattice. This converse states that any distributive lattice-homomorphic image of a lattice L is isomorphic with a ring of subsets of some collection of prime dual ideals of L . O. Frink.

Funayama, Nenosuke. Notes on lattice theory. IV. On partial (semi-) lattices. Bull. Yamagata Univ. (Nat. Sci.) 2 (1953), 171-184. (Japanese summary)

By a partial lattice the author means a set of elements such that the join and meet of a pair of elements are not necessarily always defined, but when they are defined they obey the usual laws of lattice theory. Similarly, a partial (upper) semi-lattice involves only the join operation $a \cup b$, which obeys the usual rules whenever it is defined. There is an underlying partial ordering determined by the operations. Joins and meets when they are defined are least upper bounds and greatest lower bounds in this ordering, but the least upper bound of a pair of elements may exist when the join is not defined.

An extensive theory of these systems is given. The author defines such concepts as homomorphism and strong homomorphism, ideal, dual ideal, prime ideal, imbedding and strong imbedding, strong partial lattice and strong partial semi-lattice. Some sample theorems are: A partial lattice can be strongly imbedded in a lattice if and only if it is strong. All the homomorphisms and all the strong homomorphisms on a partial lattice or partial semi-lattice form a complete lattice. The free distributive lattice generated by a partial semi-lattice is isomorphic with the ring of all J -closed subsets with finite dual crowns of the semi-lattice of all ideals generated by finite subsets. These results are illustrated by numerous examples. O. Frink (University Park, Pa.).

See also: Funayama, p. 286; Anderson, p. 287.

Rings, Fields, Algebras

Blair, R. L. A note on f -regularity in rings. Proc. Amer. Math. Soc. 6 (1955), 511-515.

An element a of a ring A is called f -regular if $(a) = (a)^2$, where (a) is the two-sided ideal generated by a . A ring is

f -regular if each of its elements is f -regular. Denote by $F(A)$ the set of all f -regular elements of A . In a previous paper [Trans. Amer. Math. Soc. 75 (1953), 136-153; MR 15, 4] the author has shown inter alia that $F(A)$ is an f -regular ideal and that $F(A/F(A)) = 0$. In the present note he rounds out the radical-like properties of $F(A)$ by showing that if B is an ideal of A then $F(B) = B \cap F(A)$ and that if A_n denotes the complete matrix ring of order n over A , then $F(A_n) = (F(A))_n$. He further shows that if $N(A)$ denotes the prime radical of A (i.e. the intersection of all prime ideals of A), then $N(A) \cap F(A) = 0$. As to the relationship of $F(A)$ to the Jacobson radical $J(A)$, he proves the equivalence of the following statements: (1) $J(A) \cap F(A) = 0$ for every ring A . (2) Every f -regular ring is semi-simple (in the sense of Jacobson). (3) Every simple ring is semi-simple. (4) For every radical ring (in the sense of Jacobson which is generated by a single element a (i.e. $A = (a)$), $A^2 \neq A$. Finally he remarks that in the presence of the descending-chain condition for right ideals, $F(A)$ coincides with the largest regular ideal of A . J. Levitzki (Jerusalem).

McCoy, Neal H. Note on subdirect sums of rings. Proc. Amer. Math. Soc. 6 (1955), 554-557.

For a given ring R and $a \in R$ put $F(a) = \{ax - x; x \in R\}$, $G(a) = ax - x + \sum (y_i a x_i - y_i x_i); x, y_i, x_i \in R$. Then $J = \{a \in R; at \in F(at) \text{ for every } t \in R\}$ is the Jacobson radical, while $N = \{a \in R; b \in G(b) \text{ for each element } b \text{ of } (a)\}$ where (a) is the two-sided ideal of R generated by a , is the Brown-McCoy radical. It is well known that $J = 0$, respectively $N = 0$, if and only if R is isomorphic to a subdirect sum of primitive rings, respectively simple rings with units. In the present note the author introduces the set $M = \{a \in R; at \in G(at) \text{ for every } t \in R\}$. It is immediate that $J \subseteq N \subseteq M$ and that $J = M = N$ if R is commutative. It follows further that M contains all nilpotent elements of R . However, in general M need not be an ideal. A subdirect sum is said to possess property P if each of the component rings is a simple ring with unit and at least one component of every nonzero element has a right inverse. A ring has property P if it is isomorphic to a subdirect sum with property P . The author shows that $M = 0$ is a necessary and sufficient condition for a ring to have property P . Some further results on subdirect sums with property P are derived and it is shown that for a regular ring R , one has $M = 0$ if and only if R is isomorphic to a subdirect sum of division rings, and that this condition is equivalent to the requirement that R should have no nonzero nilpotent elements. Finally, the author shows that if $M = 0$ and the descending chain condition holds for principal right ideals of R , then R is a discrete direct sum of division rings, and that in the presence of this chain condition the condition $M = 0$ and the requirement that R should have no nonzero nilpotent elements are equivalent. J. Levitzki (Jerusalem).

Tominaga, Hisao, and Yamada, Tetsuo. On the π -regularity of certain rings. Proc. Japan Acad. 31 (1955), 253-256.

The authors consider a property of rings: (*) to be nil and of bounded index. The transfinite chain of (*)-socles of a ring R yield a radical-like ideal $B^*(R)$. A similar chain of unions of (*)-ideals yields an ideal $U^*(R)$. If $R/U^*(R)$ is of 'locally bounded index' then it is shown that the properties: π -regularity, right (left) π -regularity and strong π -regularity of an element of R are equivalent. [Reviewer's

remark: since a (*)-ideal coincides with its lower radical, it follows that $B^*(R) = U^*(R) = \text{lower radical of } R$.

S. A. Amitsur (Jerusalem).

Kleinfeld, Erwin. Primitive alternative rings and semi-simplicity. Amer. J. Math. 77 (1955), 725-730.

Let A be an alternative ring. Define A to be primitive if it contains a regular, maximal right ideal which contains no two-sided ideal of A other than the zero ideal. The radical Q of A is defined as the intersection of its regular, maximal right ideals, and A is said to be semi-simple in case $Q=0$. Kaplansky has proved [Portugal. Math. 10 (1951), 37-50; MR 13, 8] that π -regular, primitive, alternative rings are either associative or Cayley-Dickson algebras and he conjectured that the π -regular assumption is superfluous. The present paper verifies this conjecture, and contains the following additional results. (i) If A is semi-simple, then A is a subdirect sum of primitive, alternative rings. (ii) If A is alternative, not-associative, and without proper two-sided ideals, then A is without proper one-sided ideals also. Theorem (ii) is new for nil rings and suggests again the open question: Does there exist an alternative, not-associative nil ring without proper ideals? Kaplansky's paper mentioned above shows that the Smiley radical S of A is always contained in Q and coincides with Q in the case of a Zorn ring. The author remarks that the question of whether $S=Q$ in general is still open. R. L. San Soucie (Eugene, Ore).

Northcott, D. G. A note on the $AF+B\Phi$ theorem and the theory of local rings. Proc. Cambridge Philos. Soc. 51 (1955), 545-550.

On dit qu'un anneau local Q d'idéal maximal m est "ordinaire" si l'anneau gradué associé $\sum m^n/m^{n+1}$ n'a pas d'élément nilpotent. Des raisonnements utilisant les polynômes caractéristiques et les éléments superficiels montrent que, si Q est un domaine local ordinaire de dimension 1, alors sa clôture intégrale Q' coïncide avec son premier anneau de voisinage, c'est à dire avec $Q[u^{-1}m]$ où u est un élément superficiel de degré 1 [cf. Northcott, J. London Math. Soc. 30 (1955), 360-375; MR 17, 86]. En particulier Q' est un Q -module de type fini. Soient alors v_i les valuations (en nombre fini) de centre m sur Q , et soit $e(Q)$ la multiplicité de Q ; si x et y sont deux éléments de Q tels que $v_i(y) \geq v_i(x) + e(Q) - 1$ pour tout i , alors $y \in Qx$; ceci généralise l'énoncé local du théorème $AF+B\Phi$ relatif au cas où $F=0$ a un point multiple ordinaire. P. Samuel.

Northcott, D. G. A note on classical ideal theory. Proc. Cambridge Philos. Soc. 51 (1955), 766-767.

L'auteur donne une démonstration très simple du fait que la fermeture intégrale R' d'un anneau de Dedekind R dans une extension finie de son corps des fractions est un anneau de Dedekind. Il montre, sans examiner à part le cas inséparable, que les trois propriétés classiques (intégralement clos, noethérien, tout idéal premier $\neq(0)$ est maximal) se transmettent de R à R' . La démonstration s'appuie sur le lemme suivant: soient R un anneau noethérien dont tout idéal premier $\neq(0)$ est maximal, et R' un sous anneau du corps des fractions de R qui soit un R -module de type fini; alors, pour tout élément $x \neq 0$ de R , on a $L_R(R/Rx) \geq L_{R'}(R'/R'x)$ ($L_R(E)$ désignant la longueur du R -module E). P. Samuel (Clermont-Ferrand).

Northcott, D. G. On homogeneous ideals. Proc. Glasgow Math. Assoc. 2 (1955), 105-111.

A number of known facts about homogeneous ideals in

polynomial rings are generalized to the case of an arbitrary graded commutative ring R with unit element. For example, calling an ideal in R homogeneous if it is generated by homogeneous elements, the sum and intersection of homogeneous ideals is homogeneous, the radical of a homogeneous ideal is homogeneous, and if each homogeneous ideal in R has a finite basis, then any homogeneous ideal is a finite intersection of homogeneous primary ideals and any homogeneous primary ideal has a composition series consisting of homogeneous ideals. Finally, R can be completed to a ring of infinite series (exactly as polynomial rings can be completed to rings of formal power series) and a number of relations are given between the homogeneous ideals of R and the ideals they extend to in the completion of R . M. Rosenlicht.

Flanders, Harley. Finitely generated modules. Duke Math. J. 22 (1955), 477-483.

The theory of the invariant factors of finitely generated modules over a commutative principal-ideal domain is represented in this paper in an invariant form. The proof of the existence of the invariant factor is standard, but the uniqueness theorem is obtained by a new invariant setting of the known proof of the uniqueness of the invariant factors of a matrix by the use of the g.c.d. of its minors. S. A. Amitsur (Jerusalem).

Rosenberg, Alex, and Zelinsky, Daniel. Extension of derivations in continuous transformation rings. Trans. Amer. Math. Soc. 79 (1955), 453-458.

The prototype of the results in this paper is the theorem that a derivation of a simple subalgebra into a central simple algebra can be extended to be inner. Nakayama [Duke Math. J. 19 (1952), 51-63; MR 13, 620] generalized as follows: if $A \supset B \supset C$ are simple rings with minimum condition and C is weakly Galois in A , then any derivation of B in A can be extended to be inner. The authors generalize this to the setting of continuous transformation rings; one needs additional hypotheses which are automatic in the presence of the minimum condition. A second, independent theorem is proved by similar methods. Here A is a continuous transformation ring with center Z , B is a primitive subalgebra whose division ring is finite-dimensional over Z , and other standard conditions are imposed. If it is assumed that the socle of B is contained in the socle of A , then any Z -linear derivation of B into A can be extended to be inner. If Z is perfect, the hypothesis on socles can be dropped by appealing to a theorem of Hochschild [Ann. of Math. 46 (1945), 58-67; MR 6, 114]. Thus a slight imperfection remains in the theory, the first challenging instance being that where B is a field finite-dimensional and inseparable over Z . I. Kaplansky.

Rosenberg, Alex, and Zelinsky, Daniel. Galois theory of continuous transformation rings. Trans. Amer. Math. Soc. 79 (1955), 429-452.

Galois theory began with fields (Galois), moved on to division rings (Jacobson, Cartan), simple rings with minimum condition (Hochschild, Nakayama), the ring of all linear transformations on a vector space (Dieudonné, Nakayama). The authors report that they have had access to the account of these matters in Jacobson's forthcoming Colloquium volume. They now generalize to the ring of all continuous linear transformations on a pair of dual vector spaces.

The discussion is carefully divided into two parts. First, rings are paired with rings; then the appropriate connec-

tion is made between rings and groups of automorphisms. The theorem pairing rings is proved in a broad context, specializing to the Jacobson-Bourbaki theorem in the case of a division ring. For the purpose at hand, Rosenberg's earlier results [Math. Z. 61 (1954), 150-159; MR 16, 563] would have sufficed. An interesting technical point is the serious use made of the weak topology, by working in the ring of continuous endomorphisms (not necessarily semi-linear). When the Galois theory emerges it is a one-to-one correspondence between certain subgroups and certain subrings, the restrictions in each case being by now more or less standard. The paper concludes with two natural sequels: a theorem on the extension of isomorphisms, and a condition for an intermediate ring to be normal over the bottom ring. *I. Kaplansky.*

Brown, Wm. P. An algebra related to the orthogonal group. Michigan Math. J. 3 (1955), 1-22.

Let $O(n)$ be the orthogonal group on an n -dimensional inner-product space P over a field K of characteristic 0. Let P_f be the tensor space of rank f over P , and consider the usual representation of $O(n)$ on P_f . The algebra of all linear transformations on P_f commuting with this representation of $O(n)$ is thought of as a representation of an abstract algebra ω_f^n of finite rank over K , and the structure of this algebra forms the main object of study of this paper.

R. Brauer's definition of ω_f^n by means of diagrams [Ann. of Math. (2) 38 (1937), 857-872] is first given. It is then shown that for a fixed f , ω_f^n is semisimple for sufficiently large n , a result already found, by different means, by H. Weyl [The classical groups, Princeton, 1939, chap. V, §; MR 1, 42]. If ω_f^n is semisimple, its structure is completely described as follows: $\omega_f^n = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_m$, $m = [f/2]$, \mathfrak{A}_r a two-sided ideal in ω_f^n , $\mathfrak{A}_r = \Pi_{f-2r} \otimes_K C_r$, where Π_{f-2r} is isomorphic to the group algebra of the symmetric group on $f-2r$ symbols and C_r is a total matrix algebra of degree $f!/2^r r! (f-2r)!$ over K . Furthermore, \mathfrak{A}_r is the direct sum of simple two sided ideals $A_r^{(\lambda)}$, where the summation extends over all partitions (λ) of $f-2r$, and $A_r^{(\lambda)} = \Pi^{(\lambda)} \otimes_K C_r$, with $\Pi^{(\lambda)}$ being the simple ideal of Π_{f-2r} corresponding to (λ) . In the proof use is made of an earlier result of the author's [Canad. J. Math. 7 (1955), 188-190; MR 16, 789].

In the final part of the paper the author shows that, if ω_f^n is semisimple, the invariant (under $O(n)$) subspaces P_f^x of P_f obtained by H. Weyl (loc. cit.) are given by $\varepsilon_r P_f^x$, where ε_r is the unit of the ideal \mathfrak{A}_r of ω_f^n .

A. Rosenberg (Princeton, N.J.).

Nikodým, Otton Martin, et Nikodým, Stanisława. Sur l'extension des corps algébriques abstraits par un procédé généralisé de Cantor. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17 (1954), 334-339 (1955).

Let F be an abstract algebraic field, V a linearly ordered algebraic field, and L a linearly ordered set of indices. The author considers L -sequences in F , defining fundamental and null-sequences with respect to a V -valuation in F . The latter form an ideal in the ring of fundamental L -sequences under the usual definitions of sum and product. The quotient field is called the V - L extension of F . If L has a maximum, this extension is isomorphic with F .

By the prime cardinal of cofinality of an ordered set M , without maximum, the author means the cardinal number of the uniquely determined regular initial ordinal,

α , for which there is a well-ordered subset of M , cofinal with M , and of ordinal α . The author states that, if L' and L'' have the same prime cardinal of cofinality, then the V - L' and V - L'' extensions of F are isomorphic. The V - L extension of F is therefore uniquely determined by V and the prime cardinal of cofinality of L . An L -sequence is called trivial if there is an index beyond which all terms are identical. If V has no maximum and has a prime cardinal of cofinality differing from that of L , then all fundamental L -sequences are trivial, so that the V - L extension of F is again isomorphic with F . Thus each valuation of F provides essentially only one "Cantorization".

Finally, the author announces the following theorem: If V has no maximum and has a prime cardinal of cofinality equal to the cardinal number of L , then a necessary and sufficient condition that there be at least one non-trivial fundamental L -sequence is that for every positive ϵ in V there exist an element A of F for which $0 < |A| < \epsilon$. *S. Gorn (Philadelphia, Pa.).*

Springer, T. A. Some properties of cubic forms over fields with a discrete valuation. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 512-516.

This note generalizes to cubic forms some previous results of the author on quadratic fields [same Proc. 58 (1955), 352-362; MR 17, 17]. Let K be a commutative field and E a vector space over K , of finite dimension and having basis vectors e_i . Then with each vector $x = \sum_i \xi_i e_i$ may be associated cubic forms

$$f(x) = \sum_{i,j,k} \alpha_{ijk} \xi_i \xi_j \xi_k \quad (\alpha_{ijk} \in K).$$

The form is called definite if $f(x) \neq 0$ whenever $x \neq 0$. The author proves the following. 1. If f is definite then $|f(x)|^3$ is a norm over E . 2. Suppose that the residue class field \bar{K} has the following property: There is an integer n_0 such that the dimension of any definite cubic form over \bar{K} is $\leq n_0$. Then the dimension of a definite cubic form over K is $\leq 3n_0$. (The bar over the K is omitted, an obvious misprint.) 3. If f is of dimension greater than 3 over a field K which is complete under a discrete valuation with finite residue class field, then f has a non-trivial zero in the unramified cubic extension L of K . *B. W. Jones.*

Schafer, R. D. Structure and representation of nonassociative algebras. Bull. Amer. Math. Soc. 61 (1955), 469-484.

This paper is an expository article dealing with the theory of finite-dimensional nonassociative algebras. Using the theories of finite-dimensional associative algebras and finite-dimensional Lie algebras over fields of characteristic zero as models of good behavior, the author discusses in turn the known results on the structure of arbitrary nonassociative algebras, alternative and Jordan algebras, and power associative algebras. Definitions of modules and representations for classes of non-associative algebras are given, and statements of a few more specific results on representations of alternative and Jordan algebras are included. No proofs are given. The paper contains an extensive bibliography. *C. W. Curtis.*

Andreoli, Giulio. Aritmetiche non peaniane, e loro relazioni coi numeri iterativi (coefficienti ed esponenti in un algoritmo). Ricerca, Napoli 5 (1954), no. 4, 3-12.

In this note the author discusses various algebras and the results of iterating the two binary operations. Ex-

amples are taken from the ring of residue classes modulo m , Boolean algebras, matrix algebra with Boolean elements, a certain topological algebra and the system based on $x \oplus y = xy/(x+y)$, $x \cdot y = xy$. *D. H. Lehmer.*

See also: Hochschild, p. 282; Chevalley and Tuan, p. 283; Schafer and Tomber, p. 283; Tôgô, p. 283; Karzel, p. 293; Zolcev, p. 330.

Groups, Generalized Groups

Rédei, László. Hungarian investigations in the theory of finite groups. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 315-325. (Hungarian) Expository paper.

Fuchs, László. On results of Hungarian research in the theory of infinite groups. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 327-341. (Hungarian) Expository paper.

Fuchs, L. Über die Strukturfrage der unendlichen abelschen Gruppen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 4 (1955), 91-95. Expository lecture.

Fuchs, L. On groups with finite classes of isomorphic subgroups. Publ. Math. Debrecen 3 (1954), 243-252 (1955).

A group G is called FCIS if it has finite classes of isomorphic subgroups. G is periodic and has finite conjugate classes. The author shows that a group G is FCIS if and only if it is a central extension of some subgroup of the group C of rationals modulo 1 by a finite group. This generalizes a result of Szele [Acta Math. Acad. Sci. Hungar. 3 (1952), 127-129; MR 14, 351] that groups with unitary classes of isomorphic subgroups are precisely the subgroups of C . A consequence of the proof is that the number of subgroups in the classes of isomorphic subgroups is uniformly bounded. All these classes contain the same number $k > 1$ of subgroups if and only if G is finite abelian of type (p, p) or (p, p, p) . This is a generalization of the abelian case already established by the author [Čehoslovack. Mat. Ž. 2(77) (1953), 387-390; MR 15, 682]. *F. Haimo (St. Louis, Mo.).*

Szmielew, W. Elementary properties of Abelian groups. Fund. Math. 41 (1955), 203-271.

A positive solution is obtained of the decision problem for the 'elementary theory' of abelian groups, that is, a method for deciding, of any 'arithmetical property' (=expressible in first-order calculus with equality and addition), whether it holds true in all abelian groups. The method yields a determination of all 'arithmetical classes' of abelian groups (=classes definable by an arithmetical property), and of all 'arithmetical types' (=minimal intersections of arithmetical classes). This paper presents a self-contained account of the main ideas of Tarski's theory of arithmetical classes, and of his method of 'elimination of quantifiers' [cf. Tarski, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, pp. 705-720; A decision method for elementary algebra and geometry, RAND, Santa Monica, Calif., 1948; MR 13, 521; 10, 499] upon which the present work is based. In addition to the

results mentioned, to which we shall confine our attention, various special results are proved concerning independence modulo m in abelian groups, as well as certain metamathematical consequences of the main theorems. The treatment is essentially self-contained, and avoids any dependence upon metamathematics. (In this review, for simplicity, we revert to the intuitive concepts of predicate and property.)

For each integer m , three kinds of rank for an abelian group A are defined. Consider the conditions:

- I. $\sum_1^r m_i x_i = 0$;
- II. $\sum m_i x_i = 0 \pmod{m}$;
- III. $m_i x_i = 0$ for all i ;
- IV. $m_i = 0 \pmod{m}$ for all i .

Then the rank ρ^1, ρ^2, ρ^3 , respectively of A is the maximum number r (or ∞ if none exists) of elements x_1, \dots, x_r in A such that, for all integers m_1, \dots, m_r , I & III \rightarrow IV; II \rightarrow IV; II & III \rightarrow IV. Property $R^i(m, n)$ asserts that $\rho^i \geq n$ ($i=1, 2, 3$); $K(n)$ that $nA=0$. Predicates $E(a_1, \dots, a_n)$ of arguments x_1, \dots, x_n assert that $\sum a_i x_i = 0$, and $C(m; a_1, \dots, a_n)$ that $\sum a_i x_i = 0 \pmod{m}$. In what follows, m is always a prime-power.

Let \mathcal{B} be the boolean algebra generated by the 'basic' predicates R, K, E , and C . The 'elimination of quantifiers', which forms the core of the paper, consists in exhibiting for each predicate of the form $(\exists x)F$, where F is in \mathcal{B} , an equivalent predicate F' which itself lies in \mathcal{B} . It follows that every arithmetical predicate, which is a priori expressible through predicates \mathcal{B} only by use of quantifiers, is in fact expressible directly by a predicate \mathcal{B} . The arithmetical properties of a group A , as predicates without free variables, are then given by predicates from the subalgebra C of \mathcal{B} generated by the properties R and K .

The decision problem is thus reduced to that for predicates C , for which a simple procedure is indicated, and the following theorems (5.2 and 5.3) are obtained. Two abelian groups are 'arithmetically equivalent' (=belong to the same arithmetical classes) if and only if (I) both or neither is of first kind (satisfies some $K(n)$), and (II) for all prime-powers m , and $i=1, 2, 3$, they have the same ranks. Every arithmetical type consists either of (I) all groups not of first kind with prescribed ranks, subject only to

$$(*) \quad \rho^i(p^k) = \rho^i(p^{k+1}) + \rho^3(p^k) \quad (i=1, 2);$$

or else of (II) all groups of first kind with prescribed ranks, subject to $(*)$ and the condition that all but a finite number of the $\rho^i(p^k) = 0$. *R. C. Lyndon.*

Erdős, J. On direct decompositions of torsion free abelian groups. Publ. Math. Debrecen 3 (1954), 281-288 (1955).

In a torsion-free abelian group G , elements a and b are said to be of the same type if the Prüfer heights of a and b at almost all primes are the same for both elements and if these heights are both finite for primes where they differ. The types are partly ordered in the obvious way. Let this partly ordered set satisfy the ascending chain condition. Let S be the subgroup generated by all elements whose types are beyond a fixed type α , let T be the subgroup generated by all elements whose types are at or beyond α and let U be the subgroup generated by all elements with types $\leq \alpha$. Then the author shows that G is the direct sum of subgroups, each of which has a fixed

type for all its elements, if and only if S is a direct summand of T and $S = T \cap U$. Let G have the further, more specialized property that it is the direct sum of groups of rank 1. Then all direct summands (but not necessarily all subgroups H with the property that $ng \in H$, for an integer n and for $g \in G$, implies that $g \in H$) can be decomposed into direct sums of groups of rank 1. The considerations lead to the construction of indecomposable groups of rank 2. Reference is made to the work of Baer [Duke Math. J. 3 (1937), 68–122], a portion of which is extended here.

F. Haimo (St. Louis, Mo.).

Kaloujnine, Leo. Zum Problem der Klassifikation der endlichen metabelschen p -Gruppen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 4 (1955), 1–7.

The author raises the question as to what may be considered a satisfactory set of numerical invariants in a classification problem. One can enumerate all the finite groups by a scheme involving an examination of numerical matrices corresponding to the group tables. The difficulty is that the natural number corresponding to a given finite group has no discernible connection with the properties of that group. By way of contrast, the integral invariants of a finite abelian group lead at once to the group structure. By a standard process [the author quotes, e.g., Mal'cev, Mat. Sb. N.S. 25(67) (1949), 347–366; MR 11, 323], one can put class 2 groups of exponent p in one-to-one correspondence with finite algebras R , over the field of p elements, for which the product of any three elements is 0 and multiplication is anti-commutative. By considering the null algebras $D = R/R^3$, he represents all such R by three-dimensional matrices $(a_i^{(j)})$ where $i, j = 1, 2, \dots, d$, the dimension of D , and where $l = 1, 2, \dots, k$, the dimension of R over the base field. The author points out that the rings R are completely determined by d and by all pairs of elements of D which have a zero product.

F. Haimo (St. Louis, Mo.).

Golovin, O. N., and Gol'dina, N. P. Subgroups of free metabelian groups. Mat. Sb. N.S. 37(79) (1955), 323–336. (Russian)

The authors consider a free metabelian group F_ρ , the metabelian product of a collection of ρ copies of the group of integers. [For definition of this product, the free product reduced by an appropriate subgroup of commutators, see Golovin, Mat. Sb. N.S. 28(70) (1951), 431–444 = Amer. Math. Soc. Transl. (2) 2 (1956), 89–115; MR 13, 105]. Such groups are universal for class-2 groups in that the latter are precisely the subgroups and the homomorphic images of the former. A class-2 group with a finite number of generators is a free metabelian group if and only if its derivative is a free abelian group of finite rank. A subgroup B of F_ρ is the product of a free metabelian group B_n of rank $n \leq \rho$ and of a free abelian group D of suitably restricted rank where the derivative of B_n is included in D and where the center of B includes D . Conversely, if B is a group with such a factorization $B = B_n D$, then any free metabelian group F_ρ with suitably large ρ includes a subgroup isomorphic to B . If ρ is non-finite and sufficiently large, if C is an arbitrary free metabelian group and if D is an arbitrary free abelian group, then F_ρ includes an isomorphic image of the direct product of C and D .

F. Haimo (St. Louis, Mo.).

Hughes, D. R. A note on difference sets. Proc. Amer. Math. Soc. 6 (1955), 689–692.

The author shows the existence of 1-difference sets

[see R. H. Bruck, Trans. Amer. Math. Soc. 78 (1955), 464–481; MR 16, 1081] for a class of not-abelian groups of countably infinite order.

L. J. Paige.

Higman, Graham. On infinite simple permutation groups. Publ. Math. Debrecen 3 (1954), 221–226 (1955).

Let G be a group of permutations of a set E and for $\alpha \in G$ let $D(\alpha) = \{x \in E \mid x\alpha \neq x\}$. A subset X of E is totally unstable with respect to α if $X\alpha X\alpha$ is empty. The author proves: (i) If to every α, β, γ ($\gamma \neq 1$) of elements of G corresponds an element ρ such that $\{D(\alpha) \cup D(\beta)\} \rho$ is totally unstable with respect to γ then the derived group G is simple. (ii) If a, b , are infinite cardinals, the necessary and sufficient condition that there exist a simple group of cardinal a with a subgroup of index b is that $b \leq a \leq 2^b$. (iii) If a is an infinite cardinal, there exists a simple group of cardinal a all of whose proper subgroups have index a .

L. J. Paige (Los Angeles, Calif.).

Neumann, B. H. Groups with finite classes of conjugate subgroups. Math. Z. 63 (1955), 76–96.

Let a group G be called FG if it is finitely generated, FC if the classes of conjugate elements of G are finite, FD if the derived group G' is finite, FIZ if the center has finite index in G , X if the classes of conjugate subgroups of G are finite and Y if every subgroup has finite index in its normal closure in G . A “ B ” before a condition means that “finite” is to be replaced by “boundedly finite”. The principal result is that

$$[FC] \supset [BFC] = [FD] = [Y] = [BY] \supset [FIZ] = [X] = [BX].$$

The author points out that earlier results now show that all these conditions coincide in the case of finitely generated groups. Parts of the principal result have been known, but the author shows for the first time that $[BX] = [X]$ and that $[BY] = [Y]$. He points out that the inclusion $[Y] \supset [X]$ is unexpected and that no direct proof is known. Early in the paper, he establishes that $[X]$, $[Y] \subset [FC]$, by stages analyzes the difference classes $[FC] - [FIZ]$, $[FC] - [FD]$ and $[FD] - [FIZ]$ and then proceeds to his main result. Among references are J. Erdős [Acta Math. Acad. Sci. Hungar. 5 (1954), 45–58; MR 16, 217] and the author [Proc. London Math. Soc. (3) 1 (1951), 178–187; MR 13, 316].

F. Haimo.

Neumann, B. H. Groups covered by finitely many cosets. Publ. Math. Debrecen 3 (1954), 227–242 (1955).

The author takes as his point of departure a result of Baer's that the center of a group G has finite index if and only if G is the set union of a finite number of abelian subgroups. Suppose that the group G can be covered by n cosets $C_i = A_i g_i$ of subgroups A_i , and let $D_n = \bigcap_{i=1}^n A_i$. Then $\Delta_n = \sup |G : D_n|$ taken over all G which can be irredundantly so covered by n cosets is bounded above by u_n^2 where $u_1 = 1$ and $u_{i+1} = u_i^2 + u_i$. More exactly, $\Delta_3 \leq 6$ (the inequality is sharp), $\Delta_4 \leq 36$, $\Delta_5 \leq 320$ and $\Delta_n \leq 4u_{n-1}^2$ ($n \geq 6$) where $w_4 = 8$, $w_{i+1} = w_i^2 + w_i$. If the cover is limited to subgroups rather than to cosets, then $\Delta_3 \leq 4$ (sharp), $\Delta_4 \leq 9$ (sharp), $\Delta_5 \leq 256$ and $\Delta_n \leq 4u_{n-3}^2$ ($n \geq 6$). By considering a compact subset of euclidean n -space E_n , he finds vectors $x = (x_1, x_2, \dots, x_n) \in E_n$ such that $1/x_i = u_i + 1$, $1 \leq i \leq n-1$, while $1/x_n = u_n$, and he derives the first result above from this lemma. The details range from the laborious to the subtle and are often analytic or enumerative. In the concluding section, the author shows that if G is covered by finitely many groups, each with finite

is covered by finitely many groups, each with finite derived group, then the derived group of G is finite. The paper is concluded by the result that a group G can be covered by a finite number of cyclic groups if and only if G is finite or cyclic. The reference which seems to be most closely related to the paper is the author's [J. London Math. Soc. 2y (1954), 236-248; MR 15, 931]. *F. Haimo.*

Berman, S. D. Group algebras of abelian extensions of finite groups. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 431-434. (Russian)

Let K be an algebraically closed field, G a finite group of order not divisible by the characteristic of K . Suppose that G is an abelian extension of a group H and consider the group rings $R(G, K)$ and $R(H, K)$ of the groups G and H over K . In Theorem 1 explicit formulas are given expressing the minimal central idempotents of $R(G, K)$ in terms of the central idempotents of $R(H, K)$. In Theorem 2 a complete system of minimal idempotents of $R(G, K)$ is given in terms of such a system for $R(H, K)$, provided G/H is cyclic. The author states that with the help of Theorem 1 the following results may be obtained. Theorem 3: If G/H is cyclic of order n , C_1, \dots, C_s the classes of conjugate elements of G contained in H , h_i the number of classes of H in which the class C_i may be split, then the number of classes in G is equal to $n(1/h_1 + \dots + 1/h_s)$. Theorem 4: If G is a solvable group, n the lowest common multiple of the orders of the elements of G , Π the prime field of K , then any representation of G in K is equivalent to a representation in the field $\Pi(e)$, where e is a primitive n th root of unity.

J. Levitzki (Jerusalem).

Čunišin, S. A. On Π -solvable subgroups of finite groups. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 377-378. (Russian)

The principal theorem achieves a unification and generalization of earlier results, since the special cases $h=1$ and $s=1$ have appeared [same Dokl. (N.S.) 59 (1948), 443-445; 97 (1954), 977-980; MR 9, 492; 16, 331]. A divisor h of the order g of a finite group \mathcal{G} is called a whole-block divisor provided it is 1 or is a composition block of \mathcal{G} or is a product of several composition blocks of \mathcal{G} . A divisor s of g is called separable provided either $s=1$ or both $(s, g/s)=1$ and \mathcal{G} is $\Pi(s)$ -separable, where $\Pi(s)$ is the set of all distinct prime factors of s . If the order g of the finite group \mathcal{G} has a whole-block divisor h and a separable divisor s such that $(h, s)=1$, then \mathcal{G} contains at least one subgroup of order hs which, in case $s>1$, is $\Pi(s)$ -solvable.

R. A. Good (College Park, Md.).

Černikov, S. N. On complementability of Sylow Π -subgroups in certain classes of infinite groups. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 457-459. (Russian)

A subgroup A of a group G is said to be complemented in G if there exists a subgroup B such that $AB=G$ and $A \cap B=1$. A Sylow Π -subgroup (here abbreviated as SII) is a periodic subgroup of a group G which is maximal with respect to the property that the orders of each of its elements are products of primes from the collection of primes Π . An SII is said to be arithmetically closed (a.c.) in G if its index in any including subgroup is infinite or finite with no prime divisor from Π . It is a classical result of Schur that in a finite group normal a.c. subgroups are complemented. The author states that all normal SII's and all Sylow subgroups are a.c. In a

locally solvable group all SII's are a.c. In a locally normal group, the a.c. SII's are those which have an a.c. intersection with each finite normal subgroup N . In such a group an a.c. SII is complemented if and only if each such intersection above is complemented in N . In a locally normal group a normal SII is complemented. If a group is both locally normal and locally solvable, then each SII is complemented; and a locally normal group is locally solvable if and only if its Sylow subgroups are complemented. If the product of a Sylow Π -subgroup A of a group G by its centralizer is G , then A is a direct summand. In a locally finite group, if this product has finite index for a given normal SII, then the latter is complemented. Proofs are not given in the paper.

F. Haimo (St. Louis, Mo.).

Weir, A. J. The Sylow subgroups of the symmetric groups. Proc. Amer. Math. Soc. 6 (1955), 534-541.

Let G be a permutation group of degree r . By applying $\sigma \in G$ to the columns of the matrix 1_r , one obtains a matrix $M(\sigma)$. The correspondence $\sigma \rightarrow M(\sigma)$ is a faithful representation of G . For two permutation groups A, B of degrees m, n respectively, the group $A \circ B$ of all matrices of the form

$$\begin{pmatrix} A_1 & A_2 & \dots & A_n \end{pmatrix} B^*$$

where $A_i \in M(A)$ and B^* is a matrix effecting a permutation of the columns of the matrix

$$\begin{pmatrix} A_1 & A_2 & \dots & A_n \end{pmatrix},$$

corresponding to a permutation belonging to B , is called the complete product of A and B [cf. M. Krasner et L. Kaloujnine, Acta Sci. Math. Szeged 13 (1950), 208-230; 14 (1951), 39-66, 69-82; MR 14, 242]. For this multiplication one has the associative law $A \circ (B \circ C) \cong (A \circ B) \circ C$. If $S_n \cong C \circ C \circ \dots \circ C$ (n factors) where C is cyclic of order p , then S_n is a permutation group of degree p^n and order p^n , where $\alpha = 1 + p + \dots + p^{n-1}$ and so is a Sylow p -subgroup of the symmetric group of degree p^n . A Sylow p -subgroup of a symmetric group of an arbitrary order r is a direct product of the S_n . By means of a retraction of S_{n+1} onto S_n one obtains the representation of S_{n+1} as a "split extension" $S_{n+1} \cong A^n S_n$, where A^n is elementary abelian of order p^n . It is shown that the factors of the series $A_0^n \supset A_1^n \supset \dots$, where $A_0^n = A^n$, $A_1^n = (A^n, S_n)$, $A_2^n = (A^n, S_n, S_n)$, etc., are all of order p . Subgroups of S_n having the form $P = A_0^n \circ A_1^n \circ \dots \circ A_n^n$ (partition subgroups) which may be described by certain partition diagrams are introduced and their structure studied. The author's main result is that the characteristic subgroups of S_n are precisely the normal partition subgroups.

J. Levitzki (Jerusalem).

Weir, A. J. Sylow p -subgroups of the classical groups over finite fields with characteristic prime to p . Proc. Amer. Math. Soc. 6 (1955), 529-533.

In a previous paper (see above review) the author has shown how a Sylow p -subgroup S_n of the symmetric group of degree p^n ($p \neq 2$) may be expressed as the complete product $C \circ C \circ \dots \circ C$ of n cyclic groups of order p . In the present paper this result is applied to the study of the classical groups (i.e. the general linear group, the symplectic group, the unitary group and the orthogonal groups) over the finite field $GF(q)$, where $(q, p)=1$. It is shown that any Sylow p -subgroup of any of these groups

is expressible as the direct product of the "basic" subgroups $\bar{S} \cong \bar{O} \circ \bar{O} \circ \dots \circ \bar{O}$ (n factors), where \bar{O}_n is cyclic of order p^n .
J. Levitzki (Jerusalem).

★ Dieudonné, Jean. *La géométrie des groupes classiques*. Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 5. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. vii+115 pp. DM 19.60.

This volume brings together most of the modern results concerning the so-called elementary theory of the classical groups. Here the term "classical group" is used as in the author's monograph, *Sur les groupes classiques* [Hermann, Paris, 1948; MR 9, 494] and the "elementary theory" refers roughly to results which involve subgroup and homomorphisms as opposed to results concerned for example with topology, differential geometry, etc. The approach is, of course, algebraic but, as is characteristic of the author's work in this field, is strongly influenced by geometrical notions. Although many mathematicians have contributed to the subject, the bulk of the results presented here are due to the author, the main references being the monograph cited above and his paper, On the automorphisms of the classical group [Mem. Amer. Math. Soc. no. 2 (1951); MR 13, 531]. The book contains, incidentally, an excellent bibliography. The material covered falls into four chapters: Ch. I on collineations and correlations, Ch. II on the structure of the classical groups, Ch. III on geometric characterizations of the classical groups and Ch. IV on automorphisms and isomorphisms of the classical groups. Ch. I contains most of the introductory material needed for reading the book. Although detailed proofs are usually not included, the author as a rule is able to indicate the main ideas and the principles involved, so the material is quite readable. The main complication, which seems to be characteristic of the subject, is the great multiplicity of cases which apparently need to be considered in many of the proofs.

The author has communicated the following list of corrections: On page 13 replace lines 13-15 from the bottom by the sentence: Un sous-espace V de E qui n'est pas isotrope est caractérisé par le propriété que le sous-espace orthogonal V^\perp est supplémentaire de V ; mais un tel sous-espace peut contenir des droites isotropes. On dit que V est anisotrope s'il ne contient pas de droites isotropes; un tel espace est évidemment non isotrope. On page 37 replace lines 5-8 from the bottom by the sentence: Si $A = (a_{ij})$ et si $a_{ii} \neq 0$, en retranchant des lignes de A d'indice $\neq i$ des multiples à gauche de la i -ème ligne, on obtient une matrice B dont la première colonne n'a que a_{ii} comme terme $\neq 0$; on pose alors

$$\det(A) = \psi((-1)^{i+1} a_{ii}) \det(B_{ii}),$$

en désignant par B_{ii} la matrice obtenue en supprimant dans B la première colonne et la i -ème ligne. On page 107, line 16 from the bottom, read $m=n=1$ instead of $m=n=2$. On page 111 insert in the Bibliography the item: ABE, M.: [1] Projective transformation groups over non-commutative fields, Sijo-Sûgaku-Danwakai, 240 (1942). On page 114, line 13 from the top, read J. reine angew. Math, 196, instead of: Math. Zeitschrift.

C. E. Rickart (New Haven, Conn.).

Dieudonné, Jean. *Sur les multiplicateurs des similitudes*. Rend. Circ. Mat. Palermo (2) 3 (1954), 398-408 (1955).

Let E be an n -dimensional vector space over a field K not of characteristic 2 and let f be a non-degenerate

symmetric bilinear form on E . A one-to-one linear transformation of E on itself is called a "similitude" (relative to f) if $f(u(x), u(y)) = \lambda(u)f(x, y)$, where $\lambda(u)$ is an element of K^* , the multiplicative group of non-zero elements of K . The group $GO_n(k, f)$ of all similitudes contains the orthogonal group $O_n(k, f)$ as a normal subgroup consisting of all $u \in GO_n$ with $\lambda(u) = 1$. The quotient group GO_n/O_n is isomorphic to the subgroup $M(f)$ of K^* obtained as the image of GO_n in K^* under the mapping $u \rightarrow \lambda(u)$. In this situation the following generalization of Witt's theorem is obtained: Let V, W be two non totally isotropic subspaces of E having equal dimension and let v be a linear transformation of V on W such that $f(v(x), v(y)) = \mu f(x, y)$, for all $x, y \in V$. Then in order for there to exist an extension of V in GO_n it is necessary and sufficient that $\mu \in M(f)$. The main concern of the present paper is the group $M(f)$. If n is odd, it is known that $\lambda(u)$ is necessarily a square in K^* , so attention is restricted to even $n = 2m$. Here the study of $M(f)$ can be reduced to the case in which f is "anisotropic" (i.e. 0 is the only isotropic vector). Assume f anisotropic with discriminant Δ and denote by $N(\Delta)$ the group of norms (in K) of the non-zero elements of the extension K_1 of K obtained by adjoining an element ω such that $\omega^2 = (-1)^m \Delta$ to K . The group $M(f)$ is then a subgroup of $N(\Delta)$ and is equal to $N(\Delta)$ if $m = 1$. In general $M(f) \neq N(\Delta)$. A determination of the group $M(f)$ is given when K is an extension of finite degree over the field of rationals. Finally these results are extended to hermitian forms, in which case K has an involution $\xi \rightarrow \bar{\xi}$ and the group $GU_n(k, f)$ of similitudes contains the unitary group $U_n(k, f)$ as a normal subgroup. Here the $\lambda(u)$ all belong to K_0 the field of elements of K invariant under the involution.
C. E. Rickart (New Haven, Conn.).

Finkelstein, R. J. *Generalized beta invariants*. Nuovo Cimento (10) 1 (1955), 1104-1112.

Properties of the eight-component spin representation of the rotation group in six variables are recapitulated and used to construct certain invariants. A. H. Taub.

Hosszú, M. *Some functional equations related with the associative law*. Publ. Math. Debrecen 3 (1954), 205-214 (1955).

From the associative law $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ it is possible to derive 16 equations by changing the order of the factors in the right member. These are all reducible, however, either to the associative law itself or to one of the three additional laws $x \cdot (y \cdot z) = z \cdot (y \cdot x)$, $x \cdot (y \cdot z) = y \cdot (x \cdot z)$, or $x \cdot (y \cdot z) = (z \cdot x) \cdot y$. If $x \cdot y = F(x, y)$ is a function of two real variables defined on an interval (a, b) , these laws can be written as functional equations. For example, the associative law itself becomes $F[x, F(y, z)] = F[F(x, y), z]$, where x, y, z and $F \in (a, b)$. It has been shown by L. E. J. Brouwer [Math. Ann. 67 (1909), 246-267], that the most general continuous and strictly monotonic solution of this functional equation is $F(x, y) = f^{-1}[f(x) + f(y)]$, where $f(t)$ is an arbitrary continuous, strictly monotonic function with inverse $f^{-1}(t)$. In the present paper solutions are found for the functional equations which correspond to the other three associative laws mentioned above. For example, the most general continuous strictly monotonic solution of $x \cdot (y \cdot z) = z \cdot (y \cdot x)$ is shown to be $x \cdot y = f^{-1}[\alpha^2 f(x) + \alpha f(y) + \beta]$, where $f(t)$ is an arbitrary continuous, strictly monotonic function and α, β are arbitrary constants with $\alpha \neq 0$. Finally it is shown that the most general continuously differentiable and strictly

monotonic solutions of the functional equation

$$F[x, G(y, z)] = H[K(x, y), z] \text{ are}$$

$$F(x, y) = h[\varphi(x) + \psi(x)], \quad H(x, y) = h[g(x) + f(y)],$$

$$G(x, y) = \psi^{-1}[k(x) + f(y)], \quad K(x, y) = g^{-1}[\varphi(x) + h(y)],$$

where $f(t)$, $g(t)$, $h(t)$, $k(t)$, $\varphi(t)$ and $\psi(t)$ are arbitrary strictly monotonic functions with continuous derivatives.

D. C. Murdoch (Vancouver, B.C.).

Gluskin, L. M. Homomorphisms of unilaterally simple semigroups on groups. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 673-676. (Russian)

A semigroup (that is, a system with an associative operation) is called right simple provided it properly contains no right ideals. With the aid of those elements each of which serves as a right identity for some element, a certain subset N of a right simple semigroup G is defined and then studied. If G is a right simple semigroup, N is the kernel of a homomorphism of G onto a group. Every homomorphism ϕ of G onto a semigroup with identity is uniquely determined by its kernel; the image ϕG is a group and is a homomorphic image of the group G/N .

R. A. Good (College Park, Md.).

Gluskin, L. M. Simple semigroups with zero. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 5-8. (Russian)

A semigroup (that is, a system with an associative operation) is called simple provided it possesses only trivial homomorphisms. Let G_0 be the semigroup of order two in which every product is zero. A semigroup G with zero and different from G_0 is simple if and only if both (1) it contains no ideal except 0 and G and (2) for any two distinct elements a, b in G there exist x, y in G such that exactly one of the equations $xay=0$ and $xyb=0$ holds. Each of the four conditions is sufficient that the simple semigroup G with zero and different from G_0 be isomorphic to a certain type which is completely simple in the sense of Rees [Proc. Cambridge Philos. Soc. 37 (1941), 434-435; MR 3, 199]: (1) for every a, b, c in G , $abc=0$ and $ab \neq 0$ imply $bc=0$; (2) G contains a minimal left ideal; (3) G contains a primitive idempotent; (4) every element of G has finite order.

R. A. Good.

Kuranishi, Masatake. On the group structure in homotopy groups. Nagoya Math. J. 7 (1954), 133-144.

An axiomatic characterization is given of homotopy theory, which relies, however, as the author notes, upon reference to path spaces.

The absence of group structure in $\pi_0(A, a)$ and $\pi_1(A, B, b)$ requires an extension of the concept of group to include "sets with neutral element e ". A homomorphism is required to preserve e , and to preserve multiplication if both domain and range are groups. An isomorphism is a

one-one homomorphism. The definition of exact sequence is strengthened by requiring that every isomorphism have kernel e .

A homotopy theory consists of (1) functions $\Pi_n(A, a)$, $a \in A$, $n \geq 0$, and $\Pi_n(A, B, b)$, $b \in B \subset A$, $n \geq 1$, whose values are groups except in the lowest dimensions, where they are sets with e ; (2) boundary and relativization operators ∂ and j ; (3) functions associating with continuous maps their induced homomorphisms.

Axioms 1, 2, 3, 5 state the obvious conditions on induced homomorphisms, including the homotopy axiom and commutativity with ∂ and j . A4 is exactness. A7 asserts that $\Pi_n(a, a) = e$; A8 that $\Pi_0(A, a) = \pi_0(A, a)$ with the appropriate induced homomorphisms. The remaining A6 asserts that the inclusion map, in the path space: $(\Omega_{a,A}, \Omega_{a,a}, a) \rightarrow (\Omega_{A,A}, \Omega_{A,a}, a)$ induces an isomorphism between the groups Π_n .

It is shown that two theories satisfying the axioms are either isomorphic or antiisomorphic. The argument is by induction on dimension, starting with A8 and using A6 for the inductive step. The proof that the set isomorphism $\Pi_1(X, x) \sim \Pi_0(\Omega_{x,x}, x) \sim \Pi_1'(X, x)$ is a group isomorphism is critical. This is reduced to the case $x = S^1 V S^1$, where both Π_1 and Π_1' have a group structure, and Π_1' induces a second group structure on Π_1 . A group-theoretic lemma (see two reviews following) then completes the proof.

R. C. Lyndon (Ann Arbor, Mich.).

Iwahori, Nagayosi, and Hattori, Akira. On associative compositions in finite nilpotent groups. Nagoya Math. J. 7 (1954), 145-148.

Let $f(X, Y)$ be a word in the free group on generators X, Y , and define a composition $a \circ b = f(a, b)$. Proposition 1 (2, 3): If the composition $a \circ b$ is associative for every free group (finite nilpotent group, finite p -group) generated by two elements, then $f(X, Y)$ is one of $1, X, Y, XY, YX$. Proof of 1 is by a simple combinatorial argument; 2 then follows by a lemma of Iwasawa [Proc. Imp. Acad. Tokyo 19 (1943), 272-274; MR 7, 239], and 3 by the following analogous result: If the lower central quotients of a finitely generated group G have torsion only for powers of the prime p , then the intersection of all normal subgroups M such that G/M is a finite p -group consists solely of the identity.

R. C. Lyndon.

Morimoto, Akihiko. A lemma on a free group. Nagoya Math. J. 7 (1954), 149-150.

Proposition 1 of the paper reviewed above is proved with the hypothesis of associativity replaced by the two conditions $a \circ (a \circ b) = (a \circ a) \circ b$ and $a \circ (b \circ b) = (a \circ b) \circ b$.

R. C. Lyndon (Ann Arbor, Mich.).

See also Brown, p. 232.

THEORY OF NUMBERS

Paasche, Ivan. Beweis des Moessnerschen Satzes mittels linearer Transformationen. Arch. Math. 6 (1955), 194-199.

Still another proof of the theorem first given by A. Moessner [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 29 (1952); MR 14, 353]. This proof considers separately each of the transformations whereby any row of numbers occurring in the statement of the theorem is transformed into the following row. It turns out to be possible to combine these transformations and thus prove the theorem.

H. W. Brinkmann (Swarthmore, Pa.).

Skolem, Th. On relative Pell's equations. Bull. Soc. Math. Belg. 1954, 96-105 (1955).
Expository article.

Skolem, Th. The use of a p -adic method in the theory of diophantine equations. Bull. Soc. Math. Belg. 1954, 83-95 (1955).
Expository article.

Schinzel, André. Sur l'équation indéterminée $x^2 + l = y^2$. Bull. Soc. Roy. Sci. Liège 24 (1955), 271-274.

Subba Rao, K. Some properties of Fibonacci numbers.

I. Bull. Calcutta Math. Soc. 46 (1954), 253-257.

Let u_n be the n th Fibonacci number

$$u_0 = u_1 = 1, \quad u_{n+1} = u_n + u_{n-1}.$$

It is shown that the number of Fibonacci numbers not exceeding m is asymptotic to $\log m / \log a$ where $a = \frac{1}{2}(1 + \sqrt{5})$. The number of digits in u_n is the greatest integer in

$$1 + (n+1)\log_{10} a - \frac{1}{2}\log_{10} 5.$$

Other similar results are obtained. The work is related to, but not dependent on, an earlier paper by the author [Amer. Math. Monthly 60 (1953), 680-684; MR 15, 401].

I. Niven (Berkeley, Calif.).

Olson, F. R. Arithmetic properties of Bernoulli numbers of higher order. Duke Math. J. 22 (1955), 641-653.

The Bernoulli numbers of order k may be defined by means of

$$\frac{x^k}{(e^x - 1)^k} = \sum_{m=0}^{\infty} B_m^{(k)} \frac{x^m}{m!} \quad (|x| < 2\pi).$$

Extending results due to Carlitz [Quart. J. Math. Oxford Ser. (2) 4 (1953), 112-116; MR 14, 1064] the author establishes the following two congruences:

- (1) $B_{p+1}^{(p+1)} \equiv p^4/4 - p^3(p-1)/6 \pmod{p^5}$ (p is a prime ≥ 5);
- (2) $B_p^{(p)} \equiv -p^2(p-1)/2 + p^2 B_{p-3}/36 \pmod{p^6}$ ($p \geq 7$).

Further extensions are also obtained for the generalized Bernoulli numbers of "mixed order" defined in the joint paper of Carlitz and the author [same J. 21 (1954), 405-421; MR 15, 934].

A. L. Whiteman.

Kanold, Hans-Joachim. Über mehrfach vollkommene Zahlen. J. Reine Angew. Math. 194 (1955), 218-220.

If $n = \prod p_i^{\alpha_i}$ is a positive integer, let $p = \max p_i$, $\alpha = \max \alpha_i$; $\sigma(n)$ is the sum of the divisors of n and $Q^2(n)$ is the greatest square dividing n . The author proves: (1) Suppose $n|\sigma(n)$ and $n > 6$. Then $Q^2(n) \geq \frac{1}{2}(p+1)$ and $p > \alpha + 1$. If in addition $\alpha \not\equiv 3 \pmod{4}$, then $p \geq 2\alpha + 3$ unless $n = 672$ or $n = 30240$. (2) Let $A(x)$ denote the number of integers $n \leq x$ with $n|\sigma(n)$. Then $\lim_{x \rightarrow \infty} A(x)x^{-1} = 0$.

P. Scherk (Saskatoon, Sask.).

Schinzel, Andrzej. Sur une propriété du nombre de diviseurs. Publ. Math. Debrecen 3 (1954), 261-262 (1955).

Let $\tau(n)$ denote the number of divisors of n . Then for any two natural numbers h and m there exists a natural number $n > 1$ such that $\tau(n)/\tau(n \pm i) > m$ for $i = 1, 2, \dots, h$. The proof is elementary.

W. H. Simons.

Erdős, P. On some problems of Bellman and a theorem of Romanoff. J. Chinese Math. Soc. (N.S.) 1 (1951), 409-421. (Chinese summary)

Let $\sigma_a(n)$ denote $\sum d^a$ where the sum ranges over all divisors d of n . It is proved that if $f(x)$ is a polynomial with integral coefficients and a is any integer then there exists a constant A such that

$$\sum_{n=1}^x \sigma_{-1} f(a^n) = Ax + o(x).$$

This is a partial extension of results by R. Bellman [Duke Math. J. 17 (1950), 159-168; MR 11, 715] who treated sums $\sum \sigma_{-1}(k)$. Also there is a positive constant c such

that for $0 < s < c$

$$\lim_{a \rightarrow \infty} \frac{1}{x} \sum_{k=1}^a \sigma_{-s}(a^k - 1) = \infty.$$

For fixed a and $f(x)$ it is established that the asymptotic density of integers of the form $p + f(a^n)$ is positive. This generalizes a theorem of N. P. Romanoff [Math. Ann. 109 (1934), 668-678] who treated the case $f(x) = x$.

I. Niven (Berkeley, Calif.).

Maurer, I. Remark on multiplicative arithmetic functions.

Gaz. Mat. Fiz. Ser. A. 7 (1955), 360. (Romanian)

A proof of the following elementary theorem: Given two integers a and b , let D and M stand for their greatest common divisor and least common multiple, respectively; then, if $f(n)$ is an arithmetic function with $f(1) = 1$, $f(n)$ is multiplicative if and only if $f(a) \cdot f(b) = f(D) \cdot f(M)$ holds for arbitrary integers a and b .

E. Grosswald.

Cohen, Eckford. An extension of Ramanujan's sum.

II. Additive properties. Duke Math. J. 22 (1955), 543-550.

Let n, r, k represent integers and suppose that $r \geq 1$, $k \geq 1$. For integers a, b not both zero let $(a, b)_k$ denote the greatest common k th power divisor of a and b . The extended Ramanujan sum $c_k(n, r)$ is defined as the sum $\sum \exp(2\pi i n x / r^k)$, the summation being over all $x \pmod{r^k}$ such that $(x, r^k)_k = 1$. In an earlier paper [same J. 19 (1952), 115-129; MR 13, 823] the author proved the following orthogonality property of the ordinary Ramanujan sum $c(n, r) = c_1(n, r)$. Suppose that $d_1 | r$, $d_2 | r$ and put $S = \sum c(a, d_1) c(b, d_2)$, where the summation is over a and $b \pmod{r}$ such that $n = a + b \pmod{r}$. Then $S = rc(n, d)$ or 0 according as $d = d_1 = d_2$ or $d_1 \neq d_2$. The principal aim of this paper is to extend this relation to $c_k(n, r)$. Generalizations in several directions are obtained. The main application expresses the number of solutions of the semilinear congruence $\sum_{i=1}^k a_i x^k y_i \pmod{r^k}$ (where $(a_i, r) = 1$ for all i) in the form of a singular sum involving $c_k(n, r)$. Another application furnishes a Dirichlet-series expansion for the divisor function $\sigma_k(n, k)$ (defined to be the sum of the k th powers of the k th power divisors of n).

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, L. Some partition formulas. Tôhoku Math. J. (2) 6 (1954), 149-154.

The author derives a number of identities from the elliptic-function formulas for $\wp(u) - \wp(v)$ and $\wp'(u)$ in terms of σ -functions. A fairly typical one is

$$\frac{1+x}{1-x} - 2 \sum_{n=1}^{\infty} \frac{x^{2n}}{1+x^{2n}} (x^n - x^{-n}) = \prod_{n=1}^{\infty} \frac{(1-x^{2n})^6 (1-x^{2n})}{(1-x^{2n})^3 (1-x^{2n})^3}.$$

N. J. Fine (Philadelphia, Pa.).

Gupta, Hansraj. Partitions in general. Res. Bull. Panjab Univ. no. 67 (1955), 31-38.

Let $P_m(n)$ be the number of partitions of n into integers $a_0 = 1, a_1, a_2, \dots, a_m$. Then

$$\binom{n+m}{m} \leq P_m(n) \prod_{i=1}^m a_i \leq \binom{n+\beta_m}{m},$$

where $\beta_m = a_1 + a_2 + \dots + a_m$. If $r\beta_m = o(n)$, then

$$P_r(n) \sim \binom{n+r}{r} \prod_{i=1}^r a_i^{-1}.$$

In case a_1, a_2, \dots, a_r are the primes 2, 3, \dots, p_r and

$$r^2(\log r)^2 = O(n),$$

$$P_r(n) = \left\{ \binom{n+r}{r} + \frac{1}{2} \sum_{i=1}^r (p_i - 1) (1 + o(1)) \binom{n+r-1}{r-1} \right\} \prod_{i=1}^r p_i^{-1}.$$

N. J. Fine (Philadelphia, Pa.).

Ostmann, Hans-Heinrich. Über eine Rekursionsformel in der Theorie der Partitionen. Math. Nachr. 13 (1955), 157-160.

Let A be an arbitrary set of positive integers, and for each $a_i \in A$, let $g_i = g(a_i)$ be a positive integer. Write $G = (g_1, g_2, \dots)$, and define $q(n) = p(n, A, G)$ to be the number of partitions of n using parts a_i of g_i distinct types. Let $g_*(n) = p_*(n, A, G)$ be the corresponding partition function in which no two parts are of the same value and type. Define

$$\sigma(v, A, G) = \sum_{\substack{d|v \\ d \in A}} dg(d), \quad \sigma^*(v, A, G) = \sum_{\substack{d|v \\ d \in A}} dg(d) (-1)^{v/d+1}.$$

Then, generalizing results of W. B. Ford [Amer. Math. Monthly 38 (1931), 183-184], the author shows that for $n > 0$,

$$q(n) = \frac{1}{n} \sum_{v=1}^n \sigma(v, A, G) q(n-v),$$

$$q_*(n) = \frac{1}{n} \sum_{v=1}^n \sigma^*(v, A, G) q_*(n-v).$$

The method is the standard one of computing the logarithmic derivative of the generating function in two ways. N. J. Fine (Philadelphia, Pa.).

Freiman, G. A. An elementary method of solution of problems on the partition of numbers into an unbounded number of summands. Trudy Moskov. Mat. Obšč. 4 (1955), 113-124. (Russian)

A positive increasing divergent sequence $\{a_r\}$ is given, and the object is to obtain, by elementary methods, asymptotic formulae for

$$p(N) = \frac{q(N) - q(N-h)}{h}, \quad p_r(N) = \frac{q_r(N) - q_r(N-h)}{h},$$

where $q(N)$ is the number of solutions of

$$a_1 n_1 + a_2 n_2 + \dots + a_r n_r + \dots \leq N$$

in integers $n_i \geq 0$, $q_r(N)$ is the number of solutions with $n_i = 0$ when $i > r$, and h is a fixed member of the sequence $\{a_r\}$. The rate and regularity of growth of a_r are restricted by conditions on a twice differentiable function a_r equal to a_r at $z = r$ ($r = 1, 2, \dots$). The restrictions correspond roughly to rates of growth between r^δ and $\exp(r^{\delta-\epsilon})$ ($\delta > 0$). The formula for $p_r(N)$ is

$$1) \quad p_r(N) \sim \frac{e^E}{\sqrt{(2\pi w(1-e^{-wN})Y)}} e^T,$$

where

$$E = \lim_{s \rightarrow \infty} \left\{ \int_1^s \log a_r dz - \sum_{r=1}^{s-1} \log a_r - \frac{1}{2} \log a_s \right\}$$

and w, Y, N are functions of (r, N) defined by

$$N = \int_1^r \frac{a_r dz}{e^{wz} - 1}, \quad Y = \int_1^r \frac{a_r^2 e^{wz} dz}{(e^{wz} - 1)^2},$$

$$T = Nw - \int_1^r \log(1 - e^{-wz}) dz.$$

The formula for $p(N)$ is the formal result of replacing r everywhere by ∞ . The proof of (1) is by induction from r

to $r+1$ based on the recurrence relation

$$p_{r+1}(N) = p_r(N) + p_r(N - a_{r+1}) + p_r(N - 2a_{r+1}) + \dots$$

When r is not too large, simple inequalities are derived from this and the assumed properties of a_r , but for larger values of r the argument is supplemented by estimates of $p_r(N - \Delta)/p_r(N)$ based on the assumption that suitable inequalities for $p_r(\cdot)$ are already known. The precise meaning of (1) is not stated explicitly, and the details and logical arrangement of the second stage of the induction argument are obscure. But so far as can be judged from the context, the meaning seems to be that the ratio of the two sides of (1) tends to 1 as a Pringsheim double limit when $r, N \rightarrow \infty$. A. E. Ingham.

Freiman, G. A. Inverse problems of the additive theory of numbers. Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 275-284. (Russian)

Suppose given a positive increasing divergent sequence $\{a_r\}$, and let $n(u)$ be the number of $a_r \leq u$ and $q(N)$ the number of solutions of

$$a_1 n_1 + a_2 n_2 + \dots \leq N$$

in integers $n_i \geq 0$. The author begins by stating the theorem (i) that $\log q(u) \sim Au^\alpha$ implies $n(u) \sim Bu^\beta$ (as $u \rightarrow \infty$), where A, α, B, β are positive constants connected by

$$(1-\alpha)(1+\beta) = 1, \quad B\Gamma(1+\beta)\zeta(1+\beta) = (1-\alpha)(A\alpha)^{1+\beta};$$

and he then raises the question of strengthening hypothesis and conclusion in this theorem. He proves (ii) that, if

$$\log q(u) = Au^\alpha + O(u^{\alpha_1}) \quad (\alpha_1 < \alpha),$$

then

$$n(u) = Bu^\beta + O(u^{\beta_1}/\log u);$$

and (iii) that this conclusion cannot be improved, even if the hypothesis is replaced by

$$(1) \quad q(u) \sim A_0 u^\alpha \exp(Au^\alpha).$$

The proof of (ii) is based on the formulae

$$g(s) = \int_0^\infty e^{-su} q(u) du, \quad \log g(s) = \int_0^\infty e^{-su} d\Pi(u) \quad (s > 0),$$

where

$$g(s) = \prod_{r=1}^\infty \frac{1}{1 - e^{-sa_r}}, \quad \Pi(u) = \sum_{k=1}^\infty \frac{1}{k} n\left(\frac{u}{k}\right).$$

The main steps are (a) an Abelian inference from $q(u)$ to $g(s)$ ($u \rightarrow \infty, s \rightarrow 0+$), (b) a Tauberian inference from $\log g(s)$ to $\Pi(u)$, (c) an application of the Möbius inversion formula. In the corresponding proof of (i) a classical Tauberian theorem of Hardy and Littlewood would be appropriate under (b); in (ii) this is replaced by the known developments of this theorem by Postnikov, Freud, and Korevaar. An example constructed by Korevaar in this connection provides a model for (iii). By a similar construction the sequence $\{r^{1/\delta}\}$ is modified so as to yield a sequence $\{a_r\}$ for which $n(u) = u^\beta + O_+(u^\beta/\log u)$ while the asymptotic behaviour (1) of $q(u)$ remains undisturbed; this last point being established by means of a general theorem of the author [see the paper reviewed above]. A. E. Ingham (Cambridge, England).

Kubilyus, I. P. An analogue of A. N. Kolmogorov's theorem on Markov processes in the theory of prime numbers. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 361-363. (Russian)

Let $f(p)$, defined on the sequence of primes p , be any

function satisfying: (i) $B_n^2 = \sum_{p \leq n} f^2(p)/p \rightarrow \infty$, (ii) $\Lambda_n = \max_{p \leq n} |f(p)| = o(B_n)$, as $n \rightarrow \infty$. Put $A_n = \sum_{p \leq n} f(p)/p$ and $f_n(m) = \sum_{p|m, p \leq n} f(p)$. The distribution of values of $f_n(m)$ has been discussed by Erdős and Kac [Amer. J. Math. 62 (1940), 738-742; MR 2, 42] and more recently by the reviewer [J. London Math. Soc. 30 (1955), 43-53; MR 16, 569] under conditions less general than (i) and (ii), but the methods of both papers succeed also with (i) and (ii). Erdős and Kac grounded their method on the Central-Limit (or Liapounoff's) Theorem, and the author now derives from a generalization of this theorem by Kolmogorov [Izv. Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) 1933, 363-372] the following result about the distribution of values of the partial sums $f_n(m)$: Let $a(t)$, $b(t)$ be two real functions defined on $(0, 1)$, possessing continuous first derivatives and satisfying $a(t) < 0 < b(t)$. Then the number of integers $m \leq n$ satisfying the set of inequalities

$$a(B_n/B_n) < \{f_n(m) - A_n\}/B_n < b(B_n/B_n) \\ (p=2, 3, 5, \dots; p \leq n),$$

and the number of integers $m \leq n$ satisfying the set of inequalities

$$a(B_n/B_n) < \{f_n(m) - f_n(m+1)\}/(B_n\sqrt{2}) < b(B_n/B_n) \\ (p=2, 3, 5, \dots; p \leq n),$$

are, as $n \rightarrow \infty$, each equal to $\pi v(0, 0) + o(n)$, where $v(x, t)$ is that solution of the partial differential equation $\partial v / \partial t + \frac{1}{2} \partial^2 v / \partial x^2 = 0$ for which $v(x, 1) = 1$, $a(t) < x < b(t)$, $v(a(t), x) = 0$, $0 \leq t < 1$, $v(b(t), x) = 0$, $0 \leq t < 1$. An alternative expression for $v(0, 0)$ is stated. The proof is given in outline only.

H. Halberstam (Providence, R.I.).

See also: Fraenkel, p. 226.

Analytic Number Theory

Rankin, R. A. Van der Corput's method and the theory of exponent pairs. Quart. J. Math. Oxford Ser. (2) 6 (1955), 147-153.

Let S_0 be the set of all exponent pairs (k, l) in the sense of van der Corput which can be obtained from $(0, 1)$ by a finite number of applications of the two processes of Phillips [same J. 4 (1933), 209-225] and the obvious convexity process. The author announces without proof that the greatest lower bound of $k+l-\frac{1}{2}$ for all (k, l) in S_0 is 0.32902 13568... This shows then that no combination of the processes mentioned can give as good results as Titchmarsh's two-dimensional method in the circle problem, the divisor problem, or the problem of the order of $\zeta(\frac{1}{2}+it)$. On the other hand, this calculation does make it possible to improve the known estimates for the difference between consecutive squarefree numbers, the error term in the formula for $\sum_{n \leq x} 1$, and the error term in the formula for the number of abelian groups of order not exceeding x .

P. T. Bateman (Urbana, Ill.).

Carlitz, L. On the representation of an integer as the sum of twenty-four squares. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 504-506.

Let $r_s(n)$ be the number of representations of an integer n as the sum of s squares. The author derives Ramanujan's formula for $r_{24}(n)$, and several related formulae, in very simple fashion by equating the coefficients of two series, each an ingenious combination of elliptic theta functions. The formula was proved by Hardy using

modular functions [see "Ramanujan", Cambridge, 1940, Chap. 9; MR 3, 71], and the theory of elliptic functions was applied with success, somewhat in the manner of the present paper, by Bulygin to the more general problem of determining $r_{2s}(n)$ [Izv. Imp. Akad. Nauk (6) 8 (1914), 389-404]. There is a recent evaluation of $r_{24}(n)$ by van der Pol [Nederl. Akad. Wetensch. Proc. Ser. A. 57=Indag. Math. 16 (1954), 349-361; MR 15, 935]. H. Halberstam.

Look, C. H. On the Fourier coefficients of the function $\mathfrak{T}(\omega_1, \omega_2)$. Acta Math. Sinica 4 (1954), 113-124. (Chinese. English summary)

Under the substitutions of the full modular group the modular function

$$A(\omega_1, \omega_2) = \frac{12\varphi(\omega_1/2)}{\varphi(\omega_1/4) - \varphi(\omega_1/2)} \quad (\text{Im}(\frac{\omega_2}{\omega_1}) > 0)$$

of the 4th level (Stufe) assumes six values satisfying the equation

$$(A^2 - 3 \times 2^4)^3 - j(\omega_1, \omega_2)(A^2 - 2^6) = 0$$

[see R. Fueter, Vorlesungen über die singulären Moduln ..., Teubner, Leipzig-Berlin, 1924, p. 104 ff.]. One of these values has the expansion

$$A = \sum_{m=0}^{\infty} a_m q^m, \text{ where } a_0 = 2^3, q = e^{2\pi i \omega_2 / \omega_1}.$$

The author proves that

$$a_m = \frac{\pi}{\sqrt{m}} \sum_{\substack{0 \leq h < k \\ (h, k) = 1}} \frac{\lambda(h, k)}{k} e^{-\pi i (h' + 4mh) / 2k} I_1\left(\frac{2\pi \sqrt{m}}{k}\right).$$

Here $hh' \equiv -1 \pmod{k}$; $\lambda(h, k)$ equals 0, ± 1 , or $\mp i$ according to the residue classes of h and $k \pmod{4}$; and $I_1(z)$ is the Bessel function of order 1 of an imaginary argument.

K. Mahler (Manchester).

Fogels, È. K. On prime numbers at the beginning of an arithmetic progression. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 455-456. (Russian)

The author announces that by methods of Rodosskil [Mat. Sb. N.S. 34(76) (1954), 331-356; MR 15, 935] he has obtained the following results. 1) For any $\epsilon > 0$ there exists an absolute constant $A = A(\epsilon) > 24/\epsilon$ such that in the interval $(D^A, D^{A(1+\epsilon)})$ is situated at least one prime of the progression $Du + l$, ($u = 0, 1, 2, \dots$; $(D, l) = 1$). 2) For any $\epsilon > 0$, and for every $A > A_0(\epsilon) > 30/\epsilon$, $\pi(D^A; D, l) > D^{A(1-\epsilon)}$.

W. H. Simons (Vancouver, B.C.).

Algebraic Number Theory

★ **Mann, Henry B.** Introduction to algebraic number theory. With a chapter by Marshall Hall, Jr. The Ohio State University Press, Columbus, Ohio, 1955. vii+168 pp.

The author states in the preface that the object of this book is to introduce a reader with a general background in mathematics to the theory of algebraic numbers, expecting the reader to be familiar with fundamental concepts in modern algebra such as groups, rings, fields, etc.

The author follows the classical ideal-theoretical method and develops the fundamental theory of algebraic numbers as given, e.g., in the first two parts of Hilbert's Zahlbericht [Jber. Deutsch. Math. Verein. 4 (1897), 175-546]. The main contents are as follows: algebraic

numbers and algebraic integers, properties of ideals of algebraic number fields, in particular, the theorem on the decomposition of ideals into the products of prime ideals, Minkowski's theorem on convex sets and its applications, discriminants and differentials, Hilbert's theory of ramification of prime ideals for a Galois extension, zeta-functions and the class formula, Frobenius's theorem on the density of prime ideals and Dirichlet's theorem on the primes in an arithmetic progression. The properties of quadratic fields and cyclotomic fields are also discussed in various places in illustrating the general theory.

K. Iwasawa (Cambridge, Mass.).

Nagell, Trygve. On the representations of integers as the sum of two integral squares in algebraic, mainly quadratic fields. *Nova Acta Soc. Sci. Upsal.* (4) 15 (1953), no. 11, 73 pp.

Let α be a non-zero integer in the algebraic field Ω . If α is representable as the sum of two integral squares in Ω , α is said to be an A -number in Ω . The representation $\alpha = \xi^2 + \eta^2$, where ξ, η are integers in Ω is said to be "primitive" if the principal ideals (ξ) and (η) are relatively prime. Ω is called "simple" when the number of its ideal classes is 1.

The present paper contains an exhaustive account of A -numbers in various algebraic fields. In §§2-10 the author determines all the A -numbers in the sixteen quadratic fields $K(\sqrt{D})$ for $D = \pm 2, \pm 3, \pm 7, \pm 11, \pm 19, \pm 43, \pm 67$ and ± 163 [the case $D = -1$ was dealt with by I. Niven, *Trans. Amer. Math. Soc.* 48 (1940), 405-417; MR 2, 147, and is discussed again here]. These values of D have the property, shared by at most one other value of $|D|$, that $K(\sqrt{D})$ and $K(\sqrt{-D})$ are both simple [see H. Heilbronn and E. H. Linfoot, *Quart. J. Math. Oxford Ser. 5* (1934), 293-301], and it follows from a general theorem of Dirichlet [Werke, Bd. I, Reimer, Berlin, 1889, pp. 578-588] that the Dirichlet field $K(\sqrt{D}, \sqrt{-D})$ of the fourth degree is also simple; the author makes frequent use of this result. He announces that in a future paper he will deal also with the quadratic fields $K(\sqrt{D})$ where $D = \pm 5, \pm 13$ or ± 37 .

The number of representations of an A -number in Ω is discussed in § 11; in § 12 the author uses a result of H. Weber [Math. Ann. 20 (1882), 301-329] on the representation of primes by a quadratic form, to obtain results about the existence of A -primes in a given quadratic field, and in § 13 are determined those A -numbers in the quadratic fields discussed in §§ 2-9 which have at least one primitive representation.

The proofs are given in detail, and depend mostly on arguments in terms of congruences and quadratic residue theory.

H. Halberstam (Providence, R.I.).

Hodges, John H. Representations by bilinear forms in a finite field. *Duke Math. J.* 22 (1955), 497-509.

Let $q = p^r$, $p > 2$, and let $GF(q)$ denote the finite field of order q . In this paper the author first derives formulas for determining the number $N_r(A, B)$ of pairs U, V of matrices such that $UAV = B$, where U is $s \times m$, V is $n \times t$, A is $m \times n$ of rank r , B is $s \times t$ of rank ρ , and all matrices have elements in $GF(q)$. He also indicates a connection between N_r and bilinear forms in $GF[q, x]$. The author next obtains a variety of formulas which exhibit a connection between N_r and a certain exponential sum $H(B, z)$, which for $B = 0$ reduces to the number of $s \times t$ matrices of rank z . Finally, he gives an application of the sum $H(B, z)$ to the problem of partitioning a given matrix

into the sum of n matrices. The methods employed here are similar to those used in previous papers [Carlitz, same J. 21 (1954), 123-137; Arch. Math. 5 (1954), 19-31; Carlitz and Hodges, *Duke Math. J.* 22 (1955), 393-405; MR 15, 604, 777; 17, 130] in the treatment of the analogous problems for symmetric, Hermitian and skew-symmetric matrices, respectively.

A. L. Whiteman.

Schulz, Werner. Über Reduzibilität bei gewissen Polynomen und das Tarry-Escott'sche Problem. *Math. Z.* 63 (1955), 133-144.

In 1908 Schur raised the question of the irreducibility of polynomials in the rational field that take on prescribed values for certain integral values [Arch. Math. Phys. (3) 13 (1908), 367]. It was proved that a polynomial of degree n with highest coefficient 1 that takes on the value 1 at n integral points is certainly irreducible if $n \neq 2$ or 4. Also such a polynomial that takes on the value -1 is surely irreducible. Pólya proved that if $n = \deg f(x) \geq 17$ and $f(x)$ takes on the values $\pm p$ for n integral values of x , where p is a prime, then $f(x)$ is irreducible or the product of two irreducibles of the same degree. A. Brauer showed that it suffices to take $n \geq 7$. The author recalls these and some additional results, in particular that if $\deg f(x) = 2m \geq 8$, and $f(x)$ takes on the value $\pm p$ at $2m$ integral values of x , then, when $f(x)$ factors, either the sum or difference of the factors must be constant. The author proves the following refinement of this theorem. Let $2m \geq 8$ and

$$f(x) = A x(x - c_2) \cdots (x - c_{2m}) \pm p \quad (A > 0),$$

where $0 = c_1 < c_2 < \cdots < c_{2m}$, and assume $f(x) = g(x)h(x)$. Then

$$\begin{aligned} g(x) &= A'(x - a_1) \cdots (x - a_m) \pm (-1)^m \\ &= A'(x - b_1) \cdots (x - b_m) - (-1)^m p, \\ h(x) &= A'(x - a_1) \cdots (x - a_m) + (-1)^m p \\ &= A'(x - b_1) \cdots (x - b_m) \mp (-1)^m, \end{aligned}$$

where $A' = A^{1/2}$, $a_{2r-1} = c_{4r-3}$, $a_{2r} = c_{4r}$, $b_{2r-1} = c_{4r-2}$, $b_{2r} = c_{4r-1}$ and

$$p = A' b_1 \cdots b_m \mp 1 = A' \prod_{r=1}^m c_{4r-2} c_{4r-1} \mp 1.$$

The author next recalls the connection with the Tarry-Escott problem:

$$a_1^k + \cdots + a_m^k = b_1^k + \cdots + b_m^k \quad (k = 1, \dots, m-1).$$

He proves that if the prime q divides one of the b 's then the a 's and b 's coincide (mod q); also a new formulation of the problem is given. Finally several non-trivial solutions of the system of congruences

$$a_1^k + \cdots + a_m^k = b_1^k + \cdots + b_m^k \pmod{M} \quad (k = 1, \dots, m-1)$$

are indicated. An example is furnished when $M = p = 2m+1$, where the a 's are the quadratic residues and the b 's the non-residues of the prime p .

L. Carlitz.

Geometry of Numbers, Diophantine Approximation

Macheath, A. M., and Rogers, C. A. A modified form of Siegel's mean-value theorem. *Proc. Cambridge Philos. Soc.* 51 (1955), 565-576.

If Λ is a lattice in Euclidean n -space with determinant 1, and $\varrho(x)$ a real function defined for the points of Λ , let $\varrho(\Lambda) = \sum_{\alpha \in \Lambda, \alpha \neq 0} \varrho(x)$. In his proof of the Minkowski-Hlawka theorem, Siegel [Siegel, *Ann. of Math.* (2) 46

(1945), 340-347; MR 6, 257] defines a measure $\mu(\gamma)$ for the space of all linear transformations γ of determinant 1 and considers

$$\int_F \varrho(\gamma \Lambda_0) d\mu(\gamma) / \int_F d\mu(\gamma),$$

where Λ_0 is the lattice of all points with integer coordinates and F a fundamental region defined by the use of Minkowski's theory of reduced quadratic forms. The region F is introduced because the complete space of all γ does not have finite measure. In the present paper Siegel's region F is replaced by a region $\|\gamma\| \leq K$ where $\|\gamma\| = \sup_{|x|=1} |\gamma x|$. This region is a compact subset within the region of all γ and has a finite measure. Λ_0 is replaced by a more general discrete point set for which $\lim N(r)/V(r) = d$, where $N(r)$ denotes the number of points of Λ_0 in the sphere $|x| \leq r$ and $V(r)$ the volume of the sphere. The author's principal result is that

$$\lim_{K \rightarrow \infty} \int_{\|\gamma\| \leq K} \varrho(\gamma \Lambda_0) d\mu(\gamma) / \int_{\|\gamma\| \leq K} d\mu(\gamma) = \int \varrho(x) dx,$$

where $\varrho(x)$ is assumed to be integrable in the Riemann sense and vanishes outside a bounded region. This result can be used in place of Siegel's analogous result in the proof of the Minkowski-Hlawka theorem. D. Derry.

Rogers, C. A. The moments of the number of points of a lattice in a bounded set. Philos. Trans. Roy. Soc. London. Ser. A. 248 (1955), 225-251.

The author improves the Minkowski-Hlawka Theorem ($V_0 = 2\zeta(n)$) for $n \geq 6$ to $V_0 = 2 + 2/(3[1 + 633 \times 2^{-n}])$: If a bounded symmetric set S , not containing the origin, has Jordan content $V(S) < V_0$, there will exist an n -dimensional unimodular lattice with no point in S . The proof involves defining the lattices Λ generated by $A_t = (\omega(\delta_{1t}, \dots, \delta_{(n-1)t}, \alpha_t \omega^{-n}))$ ($1 \leq t \leq n; \alpha_t = 1$), and then, with ϱ equal to the number of lattice points of Λ in S , defining the moment $\mu_n = \lim_{m \rightarrow \infty} \int_0^1 \dots \int_0^1 \varrho^m d\alpha_1 \dots d\alpha_{n-1}$, as suggested by the invariant measure of Siegel [Ann. of Math. (2) 46 (1945), 340-347; MR 6, 257]. For if S always has two lattice points of Λ_1 , the discriminant inequality may be applied to the form $\int_0^1 \dots \int_0^1 (\varrho - 2)(\xi \varrho + \eta)^2 d\alpha_1 \dots d\alpha_{n-1}$. The proof further involves explicit formulas for μ_n , in which the "principal" term is roughly $V(S)^n$ and the "error" terms are estimated by decomposing Λ into generating sub-lattices. H. Cohn (Detroit, Mich.).

Yüh, Ming-I. A note on Diophantine inequality with prime unknown. Acta Math. Sinica 3 (1953), 218-224. (Chinese. English summary)

The author proves the following analogue of a theorem of I. M. Vinogradov [Trudy Mat. Inst. Steklov. 23 (1947), Ch. V; MR 10, 599; 15, 941]. "Let $\varepsilon > 0$ be arbitrarily small; let $0 < \kappa < 1$; let P be a sufficiently large positive integer; let $f(x) = a_{n+1}x^{n+1} + \dots + a_1x$ be a polynomial with real coefficients; and let

$$a_r = \frac{a}{q} + \frac{\theta}{q^r}, \quad a \text{ and } q \text{ integers,}$$

$$(a, q) = \lambda, |\theta| < \lambda, \tau = P^{r/2}, P^\kappa \ll q \leq \tau$$

for some suffix r with $2 \leq r \leq n$. Put

$$\varrho = \begin{cases} \frac{0.041}{n^2(\log n + 2)} & \text{if } q > P^{1/4}, \\ \frac{0.37\kappa}{n^2(\log(n^2/\kappa) + 4)} & \text{if } q \leq P^{1/4}. \end{cases}$$

For every real number A there exist a prime p and an

integer v such that $|f(p) - \theta - A| < 6P^{-1/4+\varepsilon}$, $0 < p < P$. A similar theorem is stated for functions f the derivatives of which satisfy certain inequalities. The proof uses Vinogradov's methods. K. Mahler (Manchester).

Redheffer, R. M. Approximation by enumerable sets. Amer. Math. Monthly 62 (1955), 573-576.

Let $\{r_n\}$ and $\{d_n\}$ be sequences of real numbers, the latter being positive. A set E of real numbers is said to be approximated by $\{r_n\}$ to within $\{d_n\}$ if for every ξ in E , the inequality $|\xi - r_n| < d_n$ holds for infinitely many n . The author is concerned with various set-theoretic questions related to this concept; the following results are typical: (a) if $\sum d_n = \infty$, then there is a sequence $\{r_n\}$ such that every set E is approximated by $\{r_n\}$ within $\{d_n\}$; (b) if $\sum d_n < \infty$, and if there is a sequence $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$, then the measure of E is zero. W. J. LeVeque.

Roth, K. F. Rational approximations to algebraic numbers. Mathematika 2 (1955), 1-20; corrigendum, 168.

The author proves Siegel's famous conjecture that if α is an algebraic number of degree $n \geq 2$ and if $|\alpha - h/q| < q^{-k}$ for infinitely many pairs of rational integers h, q with $q > 0$, then $k \leq 2$. The history of the problem is well known: Liouville observed [C.R. Acad. Sci. Paris 18 (1844), 883-885, 910-911; J. Math. Pures Appl. 16 (1851), 133-142] that $k \leq n$; Thue proved [J. Reine Angew. Math. 135 (1909), 284-305] that $k \leq \frac{1}{2}n + 1$; Siegel proved [Math. Z. 10 (1921), 173-213] that $k \leq s + n(s+1)^{-1}$ ($s = 1, 2, \dots, n-1$), so that certainly $k \leq 2n$; Dyson proved [Acta Math. 79 (1947), 225-240; MR 9, 412] that $k \leq (2n)^{\frac{1}{2}}$. The present result is of course the best possible. The author's method consists in using, together with techniques which have been familiar to the expert for some time, the following new lemma which, although similar in appearance to lemmas used in previous attacks, provides the push needed to secure the long sought result: Let R be a polynomial in m variables X_1, \dots, X_m with rational integral coefficients all of absolute value $\leq b$; let δ be a real number with $0 < \delta < m^{-1}$, let the degree of R in X_i be $\leq r_i$ and suppose that $r_m > 10\delta^{-1}$, $r_i > r_{i+1}\delta^{-1}$ ($1 \leq i \leq m-1$); if h_i, q_i are relatively prime rational integers with $q_i > 0$, $\delta r_i \log q_i \geq \delta$, $\delta \log q_1 > m(2m+1)$, $r_i \log q_i \geq r_1 \log q_1$ ($2 \leq i \leq m$), then there exist rational integers j_1, \dots, j_m with $0 \leq j_i \leq r_i$ ($1 \leq i \leq m$) and $\sum_{i=1}^m j_i/r_i < 10^m \delta^{(1)}^m$ such that $\partial^{j_1+\dots+j_m} R(h_1/q_1, \dots, h_m/q_m) / \partial X_1^{j_1} \dots \partial X_m^{j_m} \neq 0$. The author states that his method yields also a generalization of the above theorem to a theorem on approximation to α by algebraic numbers of fixed degree. In the corrigendum he points out that this is not correct, but that a different generalisation does obtain. E. R. Kolchin.

Jarník, Vojtěch. Contribution à la théorie des approximations diophantiennes linéaires et homogènes. Czechoslovak Math. J. 4(79) (1954), 330-353. (Russian. French summary)

Let Θ be the $s \times r$ matrix with real elements Θ_{ij} ($j = 1, 2, \dots, s; i = 1, 2, \dots, r$) and put

$$\psi(t) = \chi(\Theta, t) = \min_{1 \leq i \leq s} \{ \max_{1 \leq j \leq r} |\Theta_{ji}x_j + \dots + \Theta_{ji}x_r + x_{r+j}| \},$$

where the minimum is taken over all systems of integers x_1, x_2, \dots, x_{r+s} for which $0 < \max(|x_1|, \dots, |x_r|) \leq t$. Let $\alpha = \alpha(\Theta)$ or $\beta = \beta(\Theta)$ denote the upper bound of all real γ for which

$$\limsup_{t \rightarrow \infty} t^\gamma \psi(t) < \infty \quad \text{or} \quad \liminf_{t \rightarrow \infty} t^\gamma \psi(t) < \infty,$$

respectively. Ignoring the case when $\psi(t)=0$ for all sufficiently large t , we have $r/s \leq \alpha \leq \beta \leq \infty$. The author obtains a lower bound for β as a function of α . He proves, in particular, that if $r=2$, $s \geq 1$, $\alpha < +\infty$, then $\beta \geq \alpha(\alpha-1)$, and that for $r > 2$, $s \geq 1$, $\beta \geq \alpha^{r/(r-1)} - 3\alpha$ if $(5r^2)^{r-1} < \alpha < +\infty$. He also proves that if $r=1$, $s \geq 2$, $m > 2$, $m^{s-1} > m^{s-2} + \sum_{i=0}^{s-3} m^i$ and

$$\alpha_0 = 1 - \frac{1}{n} - \dots - \frac{1}{m^{s-1}}, \quad \beta_0 = m \frac{m^{s-1} - m^{s-2} - \dots - 1}{m^{s-1} + m^{s-2} + \dots + 1},$$

then there exists a matrix Θ consisting of s linearly independent numbers for which $\alpha = \alpha_0$ and, if $m^s > 1 + 2\sum_{i=1}^{s-1} m^i$, $\beta = \beta_0$. For large m , $\beta_0 = \alpha_0^2/(1 - \alpha_0) + O(1)$, which compares with the author's general result that when Θ contains at least two linearly independent numbers, $\beta \geq \alpha^2/(1 - \alpha)$ for $\alpha < 1$ ($r=1$, $s \geq 2$). R. A. Rankin.

Jarník, Vojtěch. Contribution à la théorie métrique des fractions continues. Czechoslovak Math. J. 4(79) (1954), 318-329. (Russian. French summary)

Let p_n/q_n ($n=0, 1, 2, \dots$) be the successive convergents of the continued fraction $\theta = (a_1, a_2, a_3, \dots)$ whose partial quotients a_i are positive integers. For $\delta > 0$, let N_δ or M_δ denote the set of all numbers such that

$$\limsup_{n \rightarrow \infty} a_{n+1}/q_n^\delta > 0 \text{ or } \liminf_{n \rightarrow \infty} a_{n+1}/q_n^\delta > 0$$

respectively. Let P_∞ be the set of all numbers θ for which $\lim_{n \rightarrow \infty} a_n = +\infty$, and P_b for integral $b > 1$, the set of all θ with $a_n \geq b$ for all sufficiently large n . It is known that the Hausdorff dimension of N_δ is $2/(2+\delta)$. The author

proves that $\dim M_\delta = \frac{1}{2} \dim N_\delta$, $\dim P_\infty = \frac{1}{2}$ and that $\frac{1}{2} < \dim P_b < 1$, $\lim_{b \rightarrow \infty} \dim P_b = \frac{1}{2}$. R. A. Rankin.

Pipping, Nils. Semi-regular continued fractions. Nordisk Mat. Tidskr. 3 (1955), 96-106, 127-128. (Swedish. English summary)

The author considers semi-regular continued fractions of two special types. In the regular continued fraction, a partial quotient which is equal to 1 can be "removed" by using the identity

$$a + \frac{1}{1+1/b} = (a+1) - \frac{1}{b+1}.$$

Minkowski showed that by selecting appropriate unit partial quotients for removal the resulting semi-regular "diagonal" continued fraction has for its convergents A/B all solutions of the inequality

$$(1) \quad \left| x - \frac{A}{B} \right| < \frac{1}{2B^2}.$$

The author proposes to select for removal every other unit partial quotient in each sequence of one or more consecutive units. This gives two different semi-regular continued fractions whose convergents satisfy (1), but do not necessarily give all solutions of (1). These continued fractions are identical with the "first and second kind" fractions of Hurwitz. Among the expansions of $D^{\frac{1}{2}}$ for $D < 100$ only 6 differ from diagonal continued fractions, namely for $D=29, 53, 58, 85, 88, 97$. Symmetry of the periodic partial quotients is not preserved.

D. H. Lehmer (Berkeley, Calif.).

ANALYSIS

Clunie, J. Note on a theorem of Parthasarathy. J. London Math. Soc. 30 (1955), 511-512.

The author gives a simple proof of the following result. Let $\mu_n(x) = \inf_{h>0} \mu(x+h)/h^n$. Suppose that $\mu'(x)/\mu(x) \rightarrow \infty$ as $x \rightarrow \infty$ and that $\mu^{(n)}(x)$ is positive and increasing for $x > x_0$. Then, for every $\epsilon > 0$, $\mu_n(x) = o(\mu^{(n)}(x+\epsilon))$ as $x \rightarrow \infty$. This theorem was used by Parthasarathy to obtain a bound for $|\mu^{(n)}(x)|$ in terms of $\mu^{(n)}(r)$ when it is known that $|f(x)| < \mu(r)$, $f(z)$ an entire function of order ≥ 1 [same J. 28 (1953), 377-379; MR 14, 966]. J. Korevaar.

Theory of Sets, Functions of Real Variables

Bagemihl, F., and Gillman, L. Generalized dissimilarity of ordered sets. Fund. Math. 42 (1955), 141-165.

The paper generalizes in various directions the following problem: does there exist a (simply) ordered set E of more than one element, such that, for every pair of distinct elements a and b of E , the sets $E - \{a\}$ and $E - \{b\}$ are dissimilar? [Non-overlapping results on this same problem have also been obtained by the reviewer, Trans. Amer. Math. Soc. 79 (1955), 341-361; MR 17, 20.] The authors apply three theorems on abstract functions and decompositions of sets to derive some rather complicated results on the aforementioned problem. Only some special results will be mentioned here. A family $\{H_i\}$ of non-overlapping intervals of an ordered set H whose union is linearly dense in H is called a partition of H . For ordered sets H and M , H is said to be pseudo-similar to M if there exists a function f defined on H , with $f(H) = M$, and a partition $\{H_i\}$ of H , such that for every t ,

f acting on $i(H_i)$ is either a similarity or anti-similarity transformation. The values of f elsewhere on H are arbitrary elements of M . If f is the identity on each $i(H_i)$, then f is said to be essentially the identity. [$i(E)$ is E without its border elements.] Theorem: The linear continuum C has a decomposition into 2^{\aleph_0} mutually exclusive sets E^* having the following properties: (a) No two of these sets are anti-similar, and, no non-empty interval J^* of any E^* (including E^* itself) is anti-similar to any subset of any E^* ($\xi = \nu$ as well as $\xi \neq \nu$); further, for any $m < 2^{\aleph_0}$, J^* is not anti-similar to any subset B of the union of m of the E^* . (b) If J^* is unbordered, then neither is it similar to any such set $B \neq J^*$. (c) Excluding essentially identical mappings, J^* is not even pseudo-similar to any such set B . (d) Let A be any arbitrary subset of C that excludes less than 2^{\aleph_0} points of J^* ; then J^* may be replaced in (a) by A , in (b) by A provided that A and B meet this J^* in distinct sets, and in (c) by A provided that J^* excludes only a finite number of points of A . Results along the same line as the above theorem are given for the decomposition of C into 2^{\aleph_0} sets E^* no two of which are similar, but which upon removal of a properly chosen element from each E^* yield mutually congruent sets; and also for the decomposition of C into 2^{\aleph_0} mutually similar sets. S. Ginsburg (Hawthorne, Calif.).

Cuesta, N. On the arithmetization of the transfinite. Acta Salmant. Ser. Ci. (N.S.) 1 (1955), no. 2, 10 pp. (Spanish)

Suppose that a distinguished sequence $\{\lambda_n\}_{n<\omega}$ has been defined for every additive principal number $\lambda > \omega$ of the second number class. The object of this note is to

form a well-ordering, of type λ , of $W(\omega)$ with the aid of generalized dyadic decimals [Cuesta Dutari, *Rev. Mat. Hisp.-Amer.* (4) 3 (1943), 186-205, 242-268; MR 5, 231] and assumed well-orderings, of respective types λ_n , of $W(\omega)$.
F. Bagemihl (Notre Dame, Ind.).

Eyraud, Henri. La divisibilité asymptotique dans les suites d'ordinaux de la seconde classe. *Cahiers Rhodan.* 6 (1954), 1-7.

Let $\{\alpha_\xi\}_{\xi < \omega_1}$ and $\{\beta_\xi\}_{\xi < \omega_1}$ be two increasing sequences of ordinal numbers less than ω_1 , such that, for some $\mu < \omega_1$, $\alpha_\xi < \beta_\xi$ ($\mu \leq \xi < \omega_1$). Then there exists an increasing sequence $\{\xi_n\}_{n < \omega}$ of ordinal numbers less than ω_1 , such that $\lim_{n < \omega} \alpha_{\xi_n} = \lim_{n < \omega} \beta_{\xi_n}$. [This follows almost immediately from the existence of critical numbers of normal functions; cf. Bachmann, *Transfinite Zahlen*, Springer, Berlin-Göttingen-Heidelberg, 1955, pp. 25, 38; MR 17, 134.]
F. Bagemihl (Notre Dame, Ind.).

Eyraud, Henri. Fonctionnelles spéciales et Théorème du Continu. *Ann. Univ. Lyon. Sect. A.* (3) 17 (1954), 5-10.
An unconvincing Lyonese attempt to prove the continuum hypothesis.
F. Bagemihl.

Bruns, Günter, und Schmidt, Jürgen. Eine filtertheoretische Formulierung der Kontinuumhypothese. *Z. Math. Logik Grundlagen Math.* 1 (1955), 91-92.

If E is a set and F a filter of subsets of E , call F unbranched if it has a totally (or well) ordered basis, and call F directly representable if there is a quasi-ordering of E whose ends (sets of the form $\{y | y \leq x\}$) form a basis of F . If E is a set of cardinal number $\geq \aleph_n$, let C_n be the filter of complements of sets in E of cardinal $< \aleph_n$. In an earlier paper [Math. Nachr. 13 (1955), 169-186; MR 17, 67] the authors showed that an infinite set E is countable if and only if C_0 is unbranched, and if and only if C_0 is directly representable. In this note the result is extended to every cardinal number of the form \aleph_{n+1} . This gives as a condition equivalent to the continuum hypothesis: In a set of cardinal number of the continuum the filter C_1 is unbranched (or directly representable).
M. M. Day.

Tugué, Tosiuyuki. Sur les fonctions qui sont définies par l'induction transfinie. *J. Math. Soc. Japan* 7 (1955), 93-122.

L'A. se sert d'un procédé inductif de Kuratowski et von Neumann [Ann. of Math. (2) 38 (1937), 521-525], l'applique à une fonction initiale, en fabrique une fonction et détermine la classe de celle-ci. Soient: N un ensemble de nombres irrationnels entre 0 et 1, et $\{f_n(x)\}$ une ω suite de fonctions réelles; si $z \in N$, soit (z_0, z_1, z_2, \dots) le développement de z en fraction continue. On définit la H -fonction (ou la H_N -fonction) relative à la base N et la suite f_n de la façon suivante [v. Kondô, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 10 (1941), 35-76; MR 9, 177]: $H_N(f_n(x)) = \sup_z (\inf_k f_{z_k}(x))$ ($k < \omega$, $z \in N$); c'est l'analogie d'une opération bien connue concernant les ensembles. F étant une famille de fonctions, soient $H_N(F)$, resp. $K_N(F)$, l'ensemble des fonctions de la forme $\sup_z \inf_k f_{z_k}(x)$ (resp., $-\sup_z \inf_k -f_{z_k}(x)$); là $f_{z_k} \in F$. Le conjugué N^* de N est défini comme un ensemble de nombres irrationnels tels que $H_N(F) = K_N(F)$. Considérons le crible binaire de Lebesgue: si $t = \sum 2^{-n} c_n$ est le développement binaire d'un $0 \leq t \leq 1$ ne contenant pas une infinité de fois le chiffre 1, et si r_1, r_2, \dots est un dénombrablement des nombres rationnels entre 0, 1, soit M_t l'ensemble des r_i vérifiant $c_i = 1$; soit ℓ le type d'ordre de M_t . Soit E l'ensemble des t

tels que M_t soit bien ordonné. Soient S, X deux espaces métriques complets séparables; soit $Q(s, x)$ une fonction réelle de Baire définie sur l'espace $S \times X$; soit $\{f_n(t)\}$ une ω -suite de transformations de Baire de E en E telles que $f_n(0) = 0$, $\overline{f_n(t)} < t$ pour $t \neq 0$; soit $\{g_n(s)\}$ une ω -suite de transformations de S en S de Baire. L'A. définit alors la fonction $\phi(t, s, x)$ dans $E \times S \times X$ en posant $\phi(0, s, x) = Q(s, x)$, $\phi(t, s, x) = \sup_{n \in N} \inf_k \phi(f_{z_k}(t), g_{z_k}(s), x)$ si $t \neq 0$. Théorème: Si N est un A -ensemble, la fonction $\phi(t, s, x)$ est projective de la classe A définie sur un CA -ensemble. Si de plus N^* est un A -ensemble, $\phi(t, s, x)$ est projective de la classe CA aussi bien que de la classe A définie sur un CA -ensemble (une fonction réelle f est dite de classe A resp. de la classe CA si pour chaque c l'ensemble des x vérifiant $f(x) > c$ soit un A -ensemble resp. CA -ensemble).

Comme une application du Théorème, l'A. définit une fonction non représentable analytiquement et qui est de la classe A aussi bien que de la classe CA .
G. Kurepa.

Mycielski, Jan. About sets with strange isometrical properties. I. *Fund. Math.* 42 (1955), 1-10.

The main theorem of the paper asserts that there exists a subset E of the sphere in three-dimensional Euclidean space such that E has the power of the continuum and such that for every countable set D , the set $E - D$ is congruent to E by rotation. The corresponding problem for the plane instead of the surface of the sphere is still unsolved. The methods are group-theoretic; the principal auxiliary fact used is that the free group on a set of generators with the power of the continuum has a faithful representation in the rotation group of the sphere. The author remarks that since a free group on more than one generator has no faithful representation in the similarity group of the plane, such methods are bound to fail in the plane. An interesting observation is that the similarity group of the plane satisfies a simple identity, which can be expressed by saying that the subgroup generated by the commutators of squares is commutative.
P. R. Halmos.

Halperin, Israel. On the Darboux property. *Pacific J. Math.* 5 (1955), 703-705.

The author proves the theorem he announced in his review [MR 15, 111] of A. Császár, C. R. Premier Congrès Math. Hongrois, Akad. Kiadó, Budapest, 1952, pp. 551-560.
T. A. Botts (Charlottesville, Va.).

Viola, Tullio. Sulle funzioni quasi continue composte. *Rend. Mat. e Appl.* (5) 14 (1955), 411-421.

A function f is called quasi-continuous on a measurable set A with respect to a measure m if for every $\varepsilon > 0$ there is a closed set C, CA with $m(A - C) < \varepsilon$ such that f is continuous on C . Then f is defined almost everywhere on A . Let $f_k (k=1, 2, \dots, n)$ be quasi-continuous functions on A . Each f_k is defined on $J_k = A - N_k$ where N_k is a zero-set; setting $\bigcup_k N_k = N$, all $f_k (k=1, 2, \dots, n)$ are defined on $J = A - N$. For $P \in J$, the set of the points Q (in R_n) with coordinates $u_k = f_k(P) (k=1, 2, \dots, n)$ is called the "coset" \bar{J} of J . According to M. Picone [see M. Picone and T. Viola, *Lezioni sulla teoria moderna dell'integrazione*, Einaudi, Torino, 1952; MR 14, 256], a continuous function φ (defined on \bar{J}) of n quasi-continuous functions on J is also quasi-continuous on J . The author now proves the theorem: In order that every quasi-continuous function φ (defined on \bar{J}) of n quasi-continuous functions on J be also quasi-continuous on J , it is necessary and sufficient that every set ECJ of positive measure has a coset \bar{E} of positive measure.
A. Rosenthal (Lafayette, Ind.).

Tarnawski, E. Continuous functions in the logarithmic-power classification according to Hölder's conditions. *Fund. Math.* 42 (1955), 11-37.

If $\omega(h)$ is positive and non-decreasing for $h > 0$, let H_ω denote the set of all bounded continuous functions f on $(-\infty, \infty)$ for which there is a constant M such that $|f(x+h) - f(x)| \leq M\omega(|h|)$ for all x . H_ω^∞ is the set of all bounded continuous f on $(-\infty, \infty)$ such that $\limsup_{h \rightarrow 0} |f(x+h) - f(x)|/\omega(|h|) = \infty$ for every x . If $\omega(h) = h^p \log h^p$, the symbols $H(\delta, \gamma)$, $H^\infty(\delta, \gamma)$ are used. A function f is said to be of type 0 if $f(x) = \sum_{n=1}^\infty a_n \phi(b_n x)$, where ϕ is periodic and not constant, $\phi \in H(1, 0)$, $a_n > 0$, $\sum a_n < \infty$, $0 < b_n < b_{n+1}$, $b_n \rightarrow \infty$. Several conditions are obtained which assure that a function of type 0 belongs to H_ω or to H_ω^∞ . Some of these bear only on $\{a_n\}$ and $\{b_n\}$ and are independent of the choice of ϕ . For instance, if $\sum a_n b_n^{\delta} (\log b_n)^{-\gamma} < \infty$, then $f \in H(\delta, \gamma)$. The existence of functions in $H_\omega \cap H_\omega^\infty$ is studied for suitable ω_1 and ω_2 . For example, a function f of type 0 is constructed, for given γ and δ ($0 \leq \delta \leq 1$), which belongs to $H(\delta, \gamma)$ and to $H^\infty(\delta, \gamma)$ for all $\gamma_1 < \gamma$; the numbers a_n and b_n in this example do not depend on ϕ . *W. Rudin.*

Wilkins, J. Ernest, Jr. The average of the reciprocal of a function. *Proc. Amer. Math. Soc.* 6 (1955), 806-815.

Given a closed real interval $[a, b]$ and real numbers α and β with $0 < \alpha < \beta$, let B be the class of all concave, monotone decreasing, real-valued functions f on $[a, b]$ for which $f(a) = \beta$ and $f(b) = \alpha$. For each f in B let $I(f)$ be the average of f over $[a, b]$ multiplied by the average of the reciprocal of f over $[a, b]$. As the culmination of a sequence of twelve lemmas it is shown, by explicit construction, that there is in class B exactly one function f_0 such that for every f in B

$$1 \leq I(f) \leq I(f_0).$$

The explicit value of $I(f_0)$ is such that this sharpens, for functions of class B , a similar inequality of Pólya and Szegő valid for functions of a much wider class.

T. A. Bots (Charlottesville, Va.).

Rosenthal, Arthur. On the continuity of functions of several variables. *Math. Z.* 63 (1955), 31-38.

L'A. dimostra alcune proposizioni sugli insiemi di punti di uno spazio reale euclideo n -dimensionale e le applica allo studio della continuità di una funzione di più variabili. Nel caso bi-dimensionale i risultati dell'A. diventano: Se M è un insieme limitato del piano e contiene infiniti punti, esistono curve convesse e differenziabili almeno una volta le quali contengono infiniti punti di M , ma non si può più dire lo stesso se a queste curve si impone di esser differenziabili due volte almeno; Se $f(x, y)$ è una funzione univoca, continua in (x_0, y_0) lungo ogni curva convessa che passi per (x_0, y_0) e sia differenziabile almeno una volta, $f(x, y)$ è superficialmente continua in (x_0, y_0) , mentre la stessa conclusione non si può più trarre senz'altro se $f(x, y)$ è continua in (x_0, y_0) lungo ogni curva convessa che passi per (x_0, y_0) e sia differenziabile almeno due volte. *G. Scorza-Dragoni.*

Marcus, S. Über einen Lehrsatz von G. P. Tolstow. *Rev. Math. Phys.* 2 (1954), 59-61 (1955).

The author makes several comments on a theorem of Tolstov [Izv. Akad. Nauk SSSR. Ser. Mat. 13 (1949), 425-446; MR 11, 167]. In particular he notes the following modification of the theorem: Let $F(x, y)$ be defined in a plane domain G and continuous on each parallel to the axes; suppose further that its upper and lower partial

derivates in x , are finite outside a non-dense F_σ -set; then every interval in G contains a portion in which $F(x, y)$ is continuous. [See also *Com. Acad. R. P. Române* 2 (1952), 5-8; MR 17, 20.] *L. C. Young (Madison, Wis.).*

Nevanlinna, Rolf. A remark on differentiable mappings. *Michigan Math. J.* 3 (1955), 53-57.

The author considers a mapping $x \rightarrow y(x)$ of n -dimensional euclidean space into itself with continuous derivative (and hence continuous jacobian). Upper and lower estimates are given for the volume of the image of the sphere $|x| < r$ under the mapping. The volume is compared with quantities measuring the dilatation of "infinitesimal ellipsoids" and with quantities measuring the degree of approximation of $y(x)$ at $x=0$ by its linear term. Explicit details cannot be given here. *R. G. Barile.*

Theory of Measure and Integration

Shields, A. Sur la mesure d'une somme vectorielle. *Fund. Math.* 42 (1955), 57-60.

Let G be a connected abelian compact topological group of second countability and let m be the Haar measure on G with $m(G)=1$. For any nonvoid $A, B \subset G$ the "vectorial sum" $A+B$ is the set of all $a+b$ with $a \in A, b \in B$. The author proves: If A, B are nonvoid m -measurable sets with $m(A)+m(B) \leq 1$, then the inner m -measure of $A+B$ is $\geq m(A)+m(B)$.

This generalization of a result due to Raikov [Mat. Sb. N.S. 5(47) (1939), 425-440; MR 1, 296] for the 1-dimensional torus space and extended by A. M. Macbeath [Proc. Cambridge Philos. Soc. 49 (1953), 40-43; MR 15, 110] to the n -dimensional torus space, is deduced from a theorem of H. H. Ostmann [J. Reine Angew. Math. 187 (1950), 183-188; MR 11, 646] on the density of the vectorial sum of two sets of integers. *H. M. Schaerf.*

Swift, George. Irregular Borel measures on topological spaces. *Duke Math. J.* 22 (1955), 427-433.

X : topological space. \mathcal{C} : family of the compact sets of X . \mathcal{B} : Boolean σ -algebra generated by \mathcal{C} . \mathcal{U} : family of the open Borel (i.e. \mathcal{B} -)sets of X . μ : Borel measure on X (defined on \mathcal{B}). μ is "outer regular" at $A \in \mathcal{B}$ if $\mu(A) = \inf \{\mu(V) : A \subset V, V \in \mathcal{U}\}$, "inner regular" at A if $\mu(A) = \sup \{\mu(C) : A \supset C, C \in \mathcal{C}\}$. In § 2 examples of irregular measures are given which are used as counterexamples to the theorems in § 3. Here is a specimen of such a theorem: Let $\{A_i\}$ ($i=1, 2, \dots$) be a sequence of \mathcal{B} -sets such that for every i, j , where $i \neq j$, there exist $V_i \in \mathcal{U}, V_j \in \mathcal{U}$ separating A_i and A_j (i.e. $A_i \subset V_i, A_j \subset V_j, V_i \cap V_j = \emptyset$), $A = \bigcup A_i$ and $\mu(A) < \infty$; then μ is outer (inner) irregular at A if and only if there exists $i=i_0$ such that μ is outer(inner) irregular at A_{i_0} . A counterexample shows the necessity of the \mathcal{U} -separation requirement for the A_i 's. § 4 deals with linear combinations of measures. Here is the main theorem: Let $\{\mu_i\}$ ($i=1, 2, \dots$) be a sequence of Borel measures on X such that $\sum \mu_i(X) < \infty$; then the Borel measure μ defined by $\mu(A) = \sum \mu_i(A)$ for any $A \in \mathcal{B}$ is outer(inner) irregular at A if and only if there exists $i=i_0$ such that μ_{i_0} is outer(inner) irregular at A . Typical for § 5 is the following theorem: Let X be a Hausdorff space, Z the set of the points $x \in X$ such that $\mu(\{x\}) > 0$ (nuclear points) and $\mu(Z) < \infty$; then the Borel measure ν defined by $\nu(A) = \mu(A) - \mu(A \cap Z)$ is outer(inner) irregular at A if and only if μ is outer(inner) irregular at A . *C. Păunc.*

Nakamura, Masahiro. A remark on the integral decomposition of a measure. *Mem. Osaka Univ. Lib. Arts Ed. Ser. B.* 1954, no. 3, 25-28.

Soient S un espace compact, φ une application continue de S sur un second espace compact T ; pour tout $t \in T$, soit m_t une mesure positive sur la "fibre" $\varphi^{-1}(t)$, de masse 1 et telle que $t \rightarrow m_t$ soit vaguement continue; alors à toute fonction x appartenant à l'algèbre A des fonctions complexes continues sur S , on peut associer la fonction $x^* \in A$ telle que $x^*(s)$ soit égale à la moyenne $\int x(u) dm_t(u)$, prise sur la fibre passant par s . On a $x^* \geq 0$ si $x \geq 0$, $x^{**} = x^*$ et $(x^*y)^* = x^*y^*$; se référant à un travail antérieur en collaboration avec T. Turumaru [*Tôhoku Math. J.* 6 (1954), 182-188; MR 16, 936] l'auteur remarque qu'inversement toute application de cette nature de A dans lui-même s'obtient de cette façon. Appliquant un théorème de Dixmier sur les W^* -sous-algèbres d'une W^* -algèbre, il montre que de cette remarque on peut déduire l'existence de la mesure quotient sur un espace de Kakutani (l'énoncé de son théorème 3 est d'ailleurs incompréhensible tel quel, car l'ensemble \mathcal{F} n'intervient pas dans la conclusion de l'énoncé!).

J. Dieudonné (Evanston, Ill.).

Ionescu Tulcea, C. T. Intégrales additives. *Com. Acad. R. P. Romine* 4 (1954), 471-477. (Romanian. Russian and French summaries)

The author considers an integration process which generalizes integrals of the Riemann type, including, for example, the integrals defined by Burkill and Kolmogoroff. The starting point is a notion of "Riemann sum" which is a function defined on an object called a groupoid (which generalizes the usual subdivisions of a given set) and with values in a uniform abelian semi-group. (See a previous paper of the author [*Acad. Repub. Pop. Romine. Stud. Cerc. Mat.* 5 (1954), 73-142; MR 16, 805] for these definitions.) The integral is then defined as a limit of a Riemann sum with respect to a given filter in the groupoid. In this paper the author studies additivity properties of the integral which are obtained by imposing additional conditions on the filters involved.

C. E. Rickart.

Enomoto, Shizu. Sur une totalisation dans les espaces de plusieurs dimensions. I. *Osaka Math. J.* 7 (1955), 69-102.

In this work the author gives a detailed development in Euclidian n -space of a totalization (D), which constitutes an extension of the classic complete total (in one-dimensional space) due to A. Denjoy. In the case of $n=2$ the author stated the results without proof in some earlier notes [*Proc. Japan. Acad.* 30 (1954), 176-179, 289-290, 437-442; MR 16, 344, 345]. The line of investigation is in accord with papers by Krzyżański [*C.R. Acad. Sci. Paris*, 198 (1934), 2058-2060], J. Ridder [*C.R. Soc. Sci. Lett. Varsovie. Cl. III.* 28 (1935), 5-16], S. Kempisty [*Fund. Math.* 27 (1936), 10-37; *Fonctions d'intervalle non additives*, Hermann, Paris, 1939; MR 1, 207], Romanovski [*Mat. Sb. N.S.* 9(51) (1941), 67-120, 281-307; MR 2, 354]. If $f(x)$ is completely totalizable on an interval I , there exists a sequence of closed sets F_j ($j=1, 2, \dots$) so that the (L) (Lebesgue) integral on F_j tends to the complete total. The (L)-measure is the metric of the totalization. The values of functions of interval, given as totals (D), are approximated by (L)-integrals. Of importance is a quite subtle investigation of properties of sequences of closed sets, whose union contains an interval of the n -space. In one-space are examined properties characterizing the complete total and arising out of the

relationship of the latter to (L)-integration. It is shown that the total (D) (in n -space) can be considered as a multiple complete total of Denjoy. *W. J. Trjitzinsky.*

Functions of a Complex Variable, Generalizations

★ **Rosenbloom, P. C.** Distribution of zeros of polynomials. Lectures on functions of a complex variable, pp. 265-285. The University of Michigan Press, Ann Arbor, 1955. \$10.00.

This is a long awaited exposition of the author's extensive work on the subject of the title, some of which dates back to his unpublished doctor's thesis of 1944. Detailed proofs are given only in some of the most important places. For a few of the other proofs the reader is referred to the following earlier papers of the author [*Den 11te Skandinaviske Matematikerkongress, Trondheim*, 1949, Tanum, Oslo, 1952, pp. 130-138; *Bull. Soc. Math. France* 79 (1951), 1-58; 80 (1952), 183-215; MR 15, 220, 233].

In the first part of the paper the author obtains estimates for the number of zeros of a polynomial $P(z) = \sum_{n \leq k} a_n z^n$ in certain domains, given an upper bound for the maximum modulus M of $P(z)$ on $|z|=1$, a lower bound for $|a_0|$ and a lower bound for one of the numbers $|a_n|$, $n \geq tk$ ($0 < t \leq 1$). Case 1: the quantity $\lambda = k^{-1} \log \{M^2/|a_0 a_k|\}$ is small. Erdős and Turán have shown that λ is a measure for the deviation of the distribution of the zeros of $P(z)$ from equi-distribution around the unit circle [*Ann. of Math.* (2) 51 (1950), 105-119; MR 11, 431]. The author proves the following potential-theoretic generalization. Let φ be a non-negative mass distribution in the plane, of total mass 1, with compact support not containing the origin, and let $u_\varphi(z) = -\int \log |z-\zeta| d\varphi(\zeta)$ be its potential. Set

$$\lambda = u_\varphi(0) - 2 \min_{|z|=1} u_\varphi(z).$$

Let R denote the annulus $r \leq |z| \leq 1/r$, and S the angle $|\arg z - \alpha| \leq \tau/2$. Then $\varphi(R) \geq 1 - \lambda/\log r^{-1}$, and

$$(*) \quad |\varphi(S) - \tau/2\pi| \leq \lambda^{1/2} \log(2 + \lambda^{-1}).$$

[The author asks whether Erdős and Turán's estimate $\lambda^{1/2}$ for (*) in the case where φ consists of a finite number of equal concentrated masses also holds in the general case. The answer is yes: restricting oneself to the crucial case where all the mass lies on the unit circle, theorem 4.3 of Ganelius [*Ark. Mat.* 3 (1954), 1-50; MR 16, 23] may be applied to obtain the sharper estimate.] Case 2. When λ is not small, comparison with equi-distribution fails. The author now uses complex-variable methods to obtain a lower bound for the fraction of the zeros of $P(z)$ which lies in a sufficiently large circle, and a potential-theoretic method (his four-circle theorem, loc. cit. above) to obtain a lower bound for the fraction of the zeros of $P(z)$ in an angle. These results are too complicated to be reproduced here.

The second part of the paper is devoted mainly to a study of the distribution of the zeros of the partial sums of power series. A point z is said to belong to the domain of boundedness B of a sequence of analytic functions $\{f_n\}$ if the sequence is analytic and uniformly bounded in some neighborhood of z . The radius of boundedness $R(z)$ is then defined as the radius of the largest open disc with center at z contained in B . Now let $f(z)$ be a (formal) power series

$\sum a_k z^k$, $s_n(z) = \sum_{k \leq n} a_k z^k$. Let $a_0 = 1$. Then it is natural to compare the absolute values of the zeros of s_n with $\rho_n = |a_n|^{-1/n}$. The author sets $f_n(z) = z^n s_n(\rho_n/z) = a_n \rho_n^n + \dots + z^n$ and $c = \liminf \{\log \rho_n / \log n\}$. Theorem: When $c < \infty$, there is a subsequence $\{g_n\}$ of $\{f_n\}$ for which $R(0) \geq \min(1, e^{-c})$. Case A: $f(z)$ has a zero or finite radius of convergence, or $f(z)$ is entire of infinite order. That is, $c \leq 0$; the above case 1 applies and shows that the zeros of the g_n are asymptotically equi-distributed around the unit circle. Case B: $f(z)$ entire of finite positive order. That is, $0 < c < \infty$; the above case 2 applies and shows that there is a positive fraction of the zeros of the g_n in every angle with vertex at the origin. These results A and B are essentially the theorems on entire functions which Carlson announced in 1924 [C.R. Acad. Sci. Paris 179 (1924), 1583-1585] and which the author proved in his thesis. Carlson published his proofs only in 1948 [Ark. Mat. Astr. Fys. 35A (1948), no. 14; MR 10, 27]. As a corollary the author obtains the following result, which was also proved by the reviewer [Duke Math. J. 18 (1951), 573-592; MR 13, 222]. When $f(z)$ is a formal power series and there is an angle with vertex at the origin containing $o(n)$ zeros of the $s_n(z)$, then $f(z)$ is an entire function of order zero. [More precise results on the growth of $f(z)$ in this case have recently been announced by Edrei.] The author finally states a very detailed and precise result on the distribution of the zeros of the functions $s_n(\rho_n z)$ in the case where $f(z)$ is an entire function of finite positive order and nice asymptotic behavior. This result generalizes Szegő's work on the zeros of the partial sums of the exponential series [S.-B. Berlin, Math. Ges. 23(1924), 50-64]. J. Korevaar (Madison, Wis.).

Kay, Alan F. Distribution of zeros of sequences of polynomials of unbounded degree. Proc. Amer. Math. Soc. 6 (1955), 571-582.

Let $\{f_n(z)\}$ be an infinite sequence of polynomials of which no two have the same degree n and for which $|f_n(z)|$, n large, has a specified behavior on a certain open set R of the complex z plane. Let R_0 be a connected subset of R such that R_0 contains the origin and $|f_n(0)|$ is asymptotically not too small compared with $\sup |f(z)|$ in R_0 . In Theorem 1 the author shows that for some subsequence of $f_n(z)$ the proportion of zeros near a finite boundary point of R_0 is bounded away from zero and that for the entire sequence the number of zeros in any bounded closed subset of R_0 is $o(n)$. In Theorem 2 he shows that any R_0 with a finite boundary point cannot contain an infinite open sector. This has an application to the sectorwise distribution of the zeros of partial sums of Taylor series with one large coefficient occurring in any but the first few terms of the sum. Such distributions had been studied by Erdős and Turán [Ann. of Math. (2) 51 (1950), 105-119; Nederl. Akad. Wetensch. Proc. 51 (1948), 1146-1154, 1262-1269; MR 11, 431; 10, 372] and by A. Dvoretzky [Ann. of Math. (2) 51 (1950), 643-696; Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 246-252; MR 11, 718; 10, 696] who however assumed that the last or next to the last term had the large coefficient. The proofs are based upon Jensen's theorem, the Normal Family Theorem and elementary inequalities. M. Marden.

Krishnaiah, P. V. On Kakeya's theorem. J. London Math. Soc. 30 (1955), 314-319.

The Kakeya theorem referred to is the one that, if all the a_k are real and if $0 < a_n < \dots < a_1 < a_0$, then all the zeros of $f(z) = a_0 + a_1 z + \dots + a_n z^n$ satisfy $|z| > 1$. The

author generalizes this theorem to infinite series with complex coefficients a_k such that $\arg(a_{k-1} - a_k) = \alpha + \theta_k$, where $|\theta_k| \leq \cos^{-1}(\cos^2 \theta)$ for some θ independent of k , $0 \leq \theta \leq \pi/2$, and where $\lim a_k = \rho e^{i(\alpha + \theta)}$ as $k \rightarrow \infty$ with ρ non-negative real and $|\beta| \leq \pi/2$. He shows that then the zeros of $f(z) = \sum_{n=0}^{\infty} a_n z^n$, if any, satisfy $|z| \geq \cos \theta$. The strict inequality holds if $(a_0 - a_1) \neq 0$ and either $(a_1 - a_2) \neq 0$ or $(a_2 - a_3) \neq 0$ and generally it holds except possibly for an enumerable set of values of θ if at least two $(a_{k-1} - a_k) \neq 0$. The proofs are fairly elementary. M. Marden (Milwaukee, Wis.).

Hyllén-Cavallius, Carl. Some extremal problems for trigonometrical and complex polynomials. Math. Scand. 3 (1955), 5-20.

Consider the class Π_n of trigonometric polynomials Φ_n with real coefficients, of order $n \geq 2$, such that $|\Phi_n(x)| \leq 1$ for all (real) x , and $\Phi_n(i\eta) = \cos \alpha$ ($0 \leq \alpha \leq \pi$). The author first determines the infimum of $\Phi_n(x)$ over Π_n for given t and α . Let T_n denote the n th Chebyshev polynomial ($T_n(\cos u) = \cos nu$), and let $a = (\cos \frac{1}{2}\alpha/n)/\cosh \frac{1}{2}t$. Then if $a|\cos \frac{1}{2}x| \geq \cos(\frac{1}{2}\pi/n)$, $m(x) = T_n(a \cos \frac{1}{2}x)$; otherwise $m(x) = -1$. The author solves the corresponding problem when $\Phi_n(i\eta) = \xi + i\eta$. He applies the results to the following problem on polynomials: Determine the class $C_n(z_0)$, $z_0 \neq 1$, of polynomials $P_n(z)$, not identically zero, with complex coefficients and degree at most n ($n \geq 2$), such that $P_n(z_0) = 0$ and $|P_n(z)|$ takes its maximum on $|z| = 1$ at $z = 1$. Let c_n be the curve whose polar equation is $\cos \frac{1}{2}\phi = \frac{1}{2}(\rho^2 + \rho^{-2}) \cos(\frac{1}{2}\pi/n)$, $-\pi/n \leq \phi \leq \pi/n$. Then if z_0 is inside c_n , $C_n(z_0)$ is empty; if z_0 is on c_n , $C_n(z_0)$ consists of the multiples of a particular (explicitly given) polynomial depending on z_0 ; if z_0 is outside c_n , there are infinitely many linearly independent polynomials in $C_n(z_0)$. Cf. the following review. R. P. Boas, Jr. (Evanston, Ill.).

Hörmander, Lars. Some inequalities for functions of exponential type. Math. Scand. 3 (1955), 21-27.

The author establishes a theorem on zeros of entire functions of exponential type and uses it to prove analogues for such functions of the inequalities for trigonometric polynomials in the paper reviewed above. Let f be an entire function of positive exponential type σ , real on the real axis and with $|f(x)| \leq 1$ (class R_σ). Let $F_\sigma(x, \alpha) = \cos(\sigma^2 x^2 + \alpha^2)^{1/2}$, $\alpha \geq 0$. Then $f - F_\sigma = 0$ or else it has a zero in every interval of the real axis where $F_\sigma(x, \alpha)$ varies between -1 and 1 , and in addition at most $2k$ zeros, where k is the smallest integer not less than α/π . For $\alpha = 0$ this reduces to a theorem of Duffin and Schaeffer [Bull. Amer. Math. Soc. 44 (1938), 236-240]. The author shows that the values that can be assumed by functions in R_σ at a given point $it \neq 0$ can be written in the form $\cos(a + ib)$, $|b| \leq \sigma|t|$. If $|b| = \sigma|t|$ the only function attaining the value is $\cos(bt^{-1}x + a)$. Let $A_\sigma(x)^2 = \sigma^2 x^2 + 2x(a + 2\pi i)b/t + \sigma^2 t^2 - b^2 + (a + 2\pi i)^2$, $v = \text{integer}$. If $f(it) = \cos(a + ib)$ and $b^2 < \sigma^2 t^2$, we have $f(x) \geq \cos A_\sigma(x)$ in the interval I , where $A_\sigma(x) \leq \pi$, with strict inequality unless $f(x) = \cos A_\sigma x$. In particular ($b = 0$), if $f \in R_\sigma$ and $f(it) = \cos a$, $t \neq 0$, $0 \leq a < \pi$, then $f(x) \geq \cos[\sigma^2(x^2 + t^2) + a^2]^{1/2}$ for $\sigma^2(x^2 + t^2) + a^2 \leq \pi^2$. The case $t = 0$ can be discussed separately: if $f \in R_\sigma$ and $f(0) = \cos c$, $0 < c \leq \pi$, then $f(u + iv) \neq 1$ for $\sigma^2(u^2 + v^2) < c^2$; but for other u, v we have $f(u + iv) = 1$ for some $f \in R_\sigma$ with $f(0) = \cos c$. R. P. Boas, Jr. (Evanston, Ill.).

Havinson, S. Ya. Extremal problems for certain classes of analytic functions in finitely connected regions. Mat. Sb. N.S. 36(78) (1955), 445-478. (Russian)

The work is concerned with functions regular and one-

valued in a domain G of the complex plane bounded by a finite number of simple closed curves of finite length. The functions $f(z)$ are of class B if bounded in G and of class E_p ($p > 0$) if $|f(z)|^p |dz|$ is bounded along curves inside and near the outer boundary of G or enclosing one of the inner boundary components. They are of class $E_{p(w), p}$ if of class E_p and such that $\int |f(x)/\varrho(x)|^p |dx| \leq 1$, where the integral is taken along the frontier of G .

An introduction discusses the history and mutual relations of investigations into extremal properties of analytic functions, starting with Landau's theorem on the coefficient sums $c_0 + c_1 + c_2 + \dots + c_n$ of functions $c_0 + c_1 z + c_2 z^2 + \dots$ bounded in $|z| < 1$.

The first chapter, of a preparatory character, cites without proof properties of the boundary values of functions bounded in mean, the representation of the function by means of these boundary values and associated properties concerning sequences of functions. Generalisations to multiply connected domains of the usual statements confined to simply connected domains are made sufficient for the main purposes of the paper.

The second chapter discusses problems of duality between functions of B and E_p and associated linear functionals which are connected with meromorphic functions having similar boundary properties. A pair of extremal problems and the corresponding extremal function is considered together. There holds for example the relation

$$\sup_{f \in B_{\varrho(x)}} \left| \int_{\Gamma} f(x) w(x) dx \right| = \inf_{\varphi \in E_1} \int_{\Gamma} \varrho(x) |w(x) - \varphi(x)| |dx|,$$

where the integrals are taken round the frontier Γ of G , $w(x)$ is an arbitrary function defined and summable on Γ , while $\varrho(x) > 0$ is defined and continuous on Γ . The extremal function $f^*(z)$ regular in G is uniquely defined apart from a factor $e^{i\alpha}$. The uniqueness of the extremal function $\varphi^*(z)$ depends on the connectivity of G and the number of zeros of $f^*(z)$. The relation

$$f^*(x) \{w(x) - \varphi^*(x)\} dx = e^{i\alpha} \varrho(x) |w(x) - \varphi^*(x)| |dx|$$

holds almost everywhere on Γ (with fixed α).

The third chapter entitled "Further properties of extremal functions" moves in the direction of greater specialisation, supposing G bounded by analytic arcs and $w(x)$ analytic. For example, if G is the circle $|z| < 1$ and if $w(x)$ is analytic on $|x| = 1$ and $\varrho(x) > 0$ is such that $\log \varrho(x)$ is harmonic on $|x| = 1$, then the extremal function $f^*(z)$ for the problem

$$\sup_{f \in B_{\varrho(x)}} \left| \int_{\Gamma} f(x) w(x) dx \right|$$

has the form

$$f^*(z) = \prod \frac{z - c_k}{1 - \bar{c}_k z} \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \varrho(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right),$$

where c_1, c_2, \dots, c_m lie in $|z| < 1$.

A. J. Macintyre (Aberdeen).

Havinson, S. Ya. Extremal problems for certain classes of functions. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 421-424. (Russian)

[Cf. the preceding review.] Nine further theorems are announced. The theory is extended for example to functions satisfying inequalities $|f(x) - a_k| \leq \varrho(x)$ on the boundary components γ_k . A canonical form for extremal functions is given on the assumption that the boundary components are analytic. Extremal problems are considered for the series $\sum_{k=0}^{\infty} c_k |z|^k$, where $c_k = \int_{\gamma_k} f(z) \overline{g_k(z)} d\sigma$.

A. J. Macintyre (Aberdeen).

Lebedev, N. A. Majorizing region for the expression

$$I = \ln \frac{z^{\lambda} f'(z)^{1-\lambda}}{f(z)^{\lambda}}$$

in the class S . Vestnik Leningrad. Univ. 10 (1955), no. 8, 29-41. (Russian)

By variational methods the author finds the precise region of variation of I (see title) for each fixed z , and $\lambda > 0$, when $f(z) = z + \dots$ is regular and univalent in $|z| < 1$. As a corollary he finds sharp upper and lower bounds for $\Re I$, and Goluzin's sharp upper bound for $\arg f'(z)$ [Mat. Sb. N.S. 1(43) (1936), 127-135]. A. W. Goodman.

Singh, S. K. On the maximum term and the rank of an entire function. Acta Math. 94 (1955), 1-11.

For an entire function $f(z)$ let $\mu(r)$ denote the maximum term and $\nu(r)$ its rank; let $\mu^{(k)}(r)$ denote the maximum term for $f^{(k)}(z)$. The author establishes several results for these quantities. For example, if $f(z)$ is of order $\rho < 1$, then $r^{\rho} \mu^{(k)}(r) / \mu(r) \rightarrow 0$ if $\rho < k(1-\rho)$. If $f(z)$ is of lower order $\lambda > 1$, then $r^{-\rho} \mu^{(k)}(r) / \mu(r) \rightarrow \infty$ if $\rho < k(\lambda-1)$. We have

$$\limsup r^{-\rho} \nu(r) + \liminf r^{-\rho} \nu(r) \leq \rho \limsup r^{-\rho} \log \mu(r).$$

There are entire functions of any positive order such that $\limsup \{\log \mu(r) / \{\nu(r) \log r\}\} = 1$, and entire functions of zero order such that the limit of this ratio is 1 (which cannot occur for functions of positive order).

R. P. Boas, Jr. (Evanston, Ill.).

Boas, R. P., Jr. Growth of analytic functions along a line. J. Analyse Math. 4 (1955), 1-28.

Soit $f(z)$ une fonction régulière et de type exponentiel $b < \pi$ dans le demi plan $x > 0$; soit $\{\lambda_n\}$ une suite régulière ($\lambda_{n+1} - \lambda_n > \delta > 0$) de densité 1 ($|\lambda_n - n| < \varepsilon(n) = o(n)$); on suppose $\varepsilon(x) \nearrow$, $\varepsilon'(x) \searrow$, $\log x = o(\varepsilon(x))$ et on pose $\varepsilon_1(x) = \varepsilon(x) \log(x \varepsilon^{-1}(x))$ et $g(x) = \varepsilon_1^{-1}(x) \log |f(x)|$; alors 1) $\limsup g(\lambda_n) < \infty$ entraîne $\limsup g(x) < \infty$, 2) $\limsup g(\lambda_n) < -B < 0$ (où $B = B(b, \varepsilon_1(x), \delta)$) entraîne $\limsup g(x) < 0$ ($x \rightarrow \infty$). Le résultat est moins fort que celui annoncé dans Proc. Nat. Acad. Sci. U.S.A. 38 (1952), 503-504 [MR 14, 155] qui reste à établir. L'auteur en fait diverses applications aux fonctions de type exponentiel dans tout le plan. Exemples: 1) $\{\lambda_n\}$ et $\{\mu_n\}$ étant deux suites régulières de densité 1 convenablement reliées, une fonction de type exponentiel $< \pi$ ne peut pas à la fois décroître très vite sur $\{\lambda_n\}$ et ne pas croître très vite sur une partie de $\{-\mu_n\}$ 2) moyennant une condition sur $\varepsilon(x)$ voisine de celle de Levinson [Gap and density theorems, Amer. Math. Soc. Colloq. Publ. v. 26, New York, 1940, chap. 8; MR 2, 180], une fonction de type exponentiel nul ne peut être bornée sur $\{\lambda_n\}$ et $\{-\lambda_n\}$ sans être constante. Comme lemmes, l'auteur établit sur $\varphi(x) = \prod (1 - x^2 \lambda_n^{-2})$ les inégalités: $\log |\varphi(x)| < A \varepsilon_1(x) + O(1)$ et $\log |\varphi'(\lambda_n)| > -A \varepsilon_1(\lambda_n) + O(1)$ ($A = \text{constante}$). J. P. Kahane (Montpellier).

Leont'ev, A. F. On completeness of a system of exponential functions in a curvilinear strip. Mat. Sb. N.S. 36(78) (1955), 555-568. (Russian)

Theorem. Let $\omega(z)$ be an entire function of exponential type such that for a set of θ everywhere dense in $|\theta| < \mu$ ($\mu > 0$)

$$\lim_{r \rightarrow \infty} \frac{\log |\omega(r e^{i\theta})|}{r} = \sigma |\sin \theta|.$$

If $\{\lambda_k\}$ ($k=1, 2, \dots$) is the set of zeros of $\omega(z)$ in $|\theta| < \mu$ and if p_k is the multiplicity of λ_k , then the set of functions

$$\{e^{\lambda_k z}, z e^{\lambda_k z}, \dots, z^{p_k-1} e^{\lambda_k z}\} \quad (k=1, 2, \dots)$$

is fundamental in the space of functions regular in $\phi(x) < 3\pi < \phi(x) + 2\pi\sigma$, where $\phi(x)$ is a continuous function $(-\infty < x < \infty)$. W. H. J. Fuchs (Ithaca, N.Y.).

Rahmanov, B. N. On the theory of univalent functions. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 369-371. (Russian)

Let K be the class of functions $f(z) = z + \dots$ which are regular and univalent in $|z| < 1$ and map $|z| < 1$ onto a convex region. Let $K+S$ be the class of functions $\Phi(z) = [f(z) + z f'(z)]/2$ where $f(z) \in K$. The author announces without proof the following: (I) If $\Phi(z) \in K+S$, then $|\arg \Phi'(z)| \leq 3 \arcsin r$, and

$$\frac{2-r}{2(1-r)^2} \geq \Re \frac{\Phi(z)}{z} \geq \begin{cases} (2+r)/2(1+r)^2, & \text{if } r \leq \frac{1}{2}, \\ (3-2r-4r^2+2r^3)/2(1-r^2)^2, & \text{if } r \geq \frac{1}{2}, \end{cases}$$

and the first two bounds are sharp. (II) If $\Phi(z) \in K+S$, the sum of the first n terms of the power series is univalent in $|z| < 1 - 3(\ln n)/n$ for $n > 10$, and has only one zero in $|z| < 1 - 2(\ln n)/n$ for $n > 8$. (III) If $f(z) \in K$ then

$$(c+1)f(e^{i\alpha}z) - f(e^{-i\alpha}z)$$

is univalent in $|z| < 1$ for each $c \geq 0$ and each α in $(0, \pi/2]$. Further, a number of results are stated for functions $\varphi(z) = z + \dots$ which are regular, univalent, and starlike in $|z| < 1$, and satisfy the additional restriction $|\arg z\varphi'(z)/\varphi(z)| < \pi/2n$ for some positive integer n .

A. W. Goodman (Lexington, Ky.).

Lohwater, A. J., Piranian, G., and Rudin, W. The derivative of a schlicht function. Math. Scand. 3 (1955), 103-106.

The following theorem is established. There exists an increasing sequence $\{n_p\}$ of positive integers such that

$$f(z) = \int_0^1 \exp \left\{ \frac{1}{2} \sum_{p=1}^{\infty} w^{n_p} \right\} dw$$

is analytic in $|z| < 1$ and is continuous and univalent in $|z| \leq 1$; the Taylor series of f converges absolutely on $|z| = 1$; and for almost all θ :

- (i) $\limsup_{r \rightarrow 1} |f'(re^{i\theta})| = +\infty$,
- (ii) $\liminf_{r \rightarrow 1} |f'(re^{i\theta})| = 0$,
- (iii) $\limsup_{r \rightarrow 1} \arg f'(re^{i\theta}) = +\infty$,
- (iv) $\liminf_{r \rightarrow 1} \arg f'(re^{i\theta}) = -\infty$.

The theorem has as its object to establish the existence of a univalent analytic function whose derivative is not of bounded characteristic. The problem of Bloch concerning the derivative of a function of bounded characteristic was answered negatively by Frostman [Kungl. Fysiogr. Sällsk. i Lund Förh. 12 (1943), 169-182; MR 6, 262]. The present result shows that the Bloch problem has a negative answer even in a very restricted subclass of univalent analytic functions. M. Heins.

Jenkins, James A. On circularly symmetric functions. Proc. Amer. Math. Soc. 6 (1955), 620-624.

A simply connected domain D in the w -plane is said to be circularly symmetric if it contains $w=0$ and if its intersection with any circle $|w|=R$ is either the whole circle, or an open arc bisected by $w=R$, or empty. Let Y be the class of all functions $w=f(z)=z+a_2z^2+\dots$, schlicht in $|z| < 1$, that map $|z| < 1$ onto a circularly symmetric domain. Clearly, the a_n must be real. It is shown

that $\text{sign } \Im(z f'(z)/f(z)) = \text{sign } \Im(z)$ must hold for $|z| < 1$, from which it follows that the map of every $|z| < r$, $r < 1$, is also circularly symmetric. Again $f(z) \in Y$ if, and only if, either $f(z) \equiv z$ or both $f(z)$ and $z f'(z)/f(z)$ are typically real. Hence the theory of the circularly symmetric functions is reduced to that of the typically real ones.

W. W. Rogosinski (Newcastle-upon-Tyne).

Bagemihl, F., and Seidel, W. A problem concerning cluster sets of analytic functions. Math. Z. 62 (1955), 99-110.

This paper is closely related to a recent paper of the authors [Math. Z. 61 (1954), 186-199; MR 16, 460]. Let $f(z)$ be a regular function in $|z| < 1$. For every $e^{i\theta}$, denote by $C(\theta)$ the radial cluster set of $f(z)$ at $e^{i\theta}$. $C(\theta)$ is a continuum (a single point will be regarded as a continuum) on the Riemann sphere. The authors treat the following general problem: How can one characterize the set of continua $\{C(\theta): 0 \leq \theta \leq 2\pi\}$, or the set of continua $\{C(\theta): \theta \in M\}$, where M is some perfect, nowhere-dense subset of the interval $0 \leq \theta \leq 2\pi$, or some F_σ of first category in this interval? To answer this question, they consider the subcontinua of the Riemann sphere as points of a suitable metric space \mathcal{L} [cf. C. Kuratowski, Topologie, t. II, Warszawa-Wrocław, 1950, p. 20; MR 12, 517]. The authors prove that if $f(z)$ is a continuous function in $|z| < 1$ and if M is an analytic set on $|z|=1$, then $\{C(\theta): e^{i\theta} \in M\}$ is an analytic set in \mathcal{L} . Further they prove that for some simple sets M there exist regular functions, $f(z)$, in $|z| < 1$, such that the corresponding set $\{C(\theta): e^{i\theta} \in M\}$ is an analytic set, but not a Borel set, in \mathcal{L} . They conjecture that if A is a non-empty analytic set in \mathcal{L} , then, if M is, e.g., a perfect, nowhere dense subset of $|z|=1$, there exists a regular function in $|z| < 1$, for which $\{C(\theta): e^{i\theta} \in M\} = A$. They show that this is true for a particular A , namely, the set of all locally connected subcontinua of the Riemann sphere. Some results in this direction are obtained. K. Noshiro (Nagoya).

Bagemihl, F., and Seidel, W. Regular functions with prescribed measurable boundary values almost everywhere. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 740-743.

Les auteurs énoncent une série de résultats fondés sur le théorème suivant: soit G une région du plan complexe limitée par des courbes de Jordan B_k ($k=1, \dots, q$) sans points communs; on définit dans G (d'une manière précisée par les auteurs) des courbes C_k et des arcs J_k balayant le voisinage de B_k dans G quand leur origine ζ parcourt C_k ; on se donne, sur chaque C_k , un F_σ de première catégorie, soit M_k , et, définies sur chaque M_k , des fonctions à valeurs réelles de classe de Baire ≤ 1 , soit $\alpha_k(\zeta)$ et $\beta_k(\zeta)$; alors il existe une fonction $f(z)$ régulière dans G , telle que, pour chaque k , $\Re f(z) \rightarrow \alpha_k(\zeta)$ et $\Im f(z) \rightarrow \beta_k(\zeta)$ quand $z \rightarrow B_k$ le long de J_k . Parmi les résultats qui dérivent de ce théorème, signalons le plus simple: étant donné deux fonctions mesurables à valeurs réelles sur $[0, 2\pi]$, soit λ et μ , il existe deux séries trigonométriques conjuguées qui convergent presque partout au sens d'Abel respectivement vers λ et vers μ . J. P. Kahane.

Lec, Ke-chun. Über die Verallgemeinerung einiger Ergebnisse der Wertverteilungstheorie der meromorphen Funktionen. Acta Math. Sinica 3 (1953), 87-100. (Chinese. German summary)

Es wird in der vorliegenden Arbeit beabsichtigt, die Methode der Ahlfors'schen Theorie der Überlagerungs-

flächen in ihrer Anwendung auf die Wertverteilungstheorie der meromorphen Funktionen weiter auszubauen um dadurch einige Resultate, die bisher nur im "Punktfall" bekannt sind, auch auf "Gebietsfall" zu verallgemeinern. Der II. Abschnitt befasst sich mit der Defektrelation im hyperbolischen Falle, wobei eine der Nevanlinnaschen ganz entsprechende Form erzielt wird. Der III. Abschnitt befasst sich mit der Millouxschen "cercles de remplissage" entsprechenden Erscheinungen bei meromorphen Funktionen nullter Ordnung im parabolischen Falle, während der IV. Abschnitt mit der Borelschen Richtung bei Funktionen endlicher Ordnung.

Author's summary.

Bochner, S. Green-Goursat theorem. Math. Z. 63 (1955), 230-242.

Nach dem Goursatschen Beweisprinzip für den Cauchyschen Integralsatz $\oint f(z) dz = 0$ wird die Integralformel

$$\int_C A dy - B dx = \int_D (A_x + B_y) dx dy$$

unter sehr allgemeinen Voraussetzungen hergeleitet: 1. D sei ein beschränktes, von endlich vielen einfach geschlossen rektifizierbaren Kurven C berandetes Gebiet der z -Ebene ($z = x + iy$). 2. Die Funktionen $A(x, y)$ und $B(x, y)$ seien in $D + C$ definiert und stetig. 3. A und B seien in jedem Punkt $z_0 \in D$ total differenzierbar:

$$A(x, y) = A(x_0, y_0) + A_{x_0}(x - x_0) + A_{y_0}(y - y_0) + o(|z - z_0|),$$

$$B(x, y) = B(x_0, y_0) + B_{x_0}(x - x_0) + B_{y_0}(y - y_0) + o(|z - z_0|).$$

4. $\Phi(x, y) = A_x + B_y$ sei über D absolut Lebesguesch integrierbar: $\int_D |\Phi| dx dy < \infty$. 5. In jedem Punkt $z_0 \in D$ sei

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \int_{|z - z_0| < \epsilon} |\Phi(x, y) - \Phi(x_0, y_0)| dx dy = 0.$$

Als eine der Anwendungen ergibt sich unter ähnlichen Voraussetzungen für komplexwertige Funktionen $f(z, \bar{z})$ die Cauchysche Integralformel mit Restglied:

$$f(z, \bar{z}) = \frac{1}{2\pi i} \int_C \frac{f(\zeta, \bar{\zeta}) d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_D \frac{f(\zeta, \bar{\zeta}) d\bar{\zeta} d\zeta}{\zeta - z}.$$

Die Ergebnisse werden auf Funktionen von n Veränderlichen, $n \geq 1$, verallgemeinert. *F. Sommer* (Münster).

Note ★ **Bochner, Salomon. Functions in one complex variable as viewed from the theory of functions in several variables.** Lectures on functions of a complex variable, pp. 315-333. The University of Michigan Press, Ann Arbor, 1955. \$10.00.

Der Verf. gibt zunächst eine Anwendung der Analysis zweier reeller Veränderlichen zum Beweis des Cauchyschen Integralsatzes ($\int_D f dz = 0$) und beweist einen ähnlichen Satz für harmonische Funktionen ($\int_D (u_x dy - u_y dx) = 0$, falls $u_{xx} + u_{yy} = 0$). Ebenso wird gezeigt, wie der Satz von Morera, dass eine komplexwertige Funktion holomorph ist, wenn $\int_D f dz = 0$ ist, mit Hilfe der reellen Funktionentheorie zweier Veränderlichen hergeleitet werden kann. Die Severische Verallgemeinerung dieses Satzes auf mehrere Veränderliche wird besprochen.

Aus einem früher vom Verf. bewiesenen Satz über Funktionen, die einem gewissen System von reellen Differentialgleichungen genügen, folgen Aussagen über die Fortsetzbarkeit von holomorphen Funktionen f in gewisse hinreichend dünne Punktmengen (z.B. in isolierte Randpunkte etc.). Schliesslich wird ein Weg angegeben,

wie man den Osgoodschen Satz, dass die Funktionaldeterminante eindeutiger holomorpher Transformationen nicht verschwindet, mit Hilfe der angegebenen Fortsetzungssätze beweist.

Am Ende der Arbeit zeigt der Verf., dass man die Existenz eines Additionstheorems für eine holomorphe Funktion einer Veränderlichen $\varphi(z)$ aus gewissen Eigenschaften der Funktion $\varphi(z, w) = \varphi(z + w)$ folgern kann. Ferner werden automorphe Formen untersucht. Der dabei verwendete Satz 9 lässt sich neuerdings unmittelbar aus dem sehr allgemein gehaltenen Satz von R. Remmert folgern, dass der Körper der meromorphen Funktionen über einer n -dimensionalen kompakten komplexen Mannigfaltigkeit eine endliche algebraische Erweiterung des Körpers der komplexen Zahlen in k Unbestimmten ($k \leq n$) ist [Dissertation, Münster, 1954]. *H. Grauert*.

Lebedev, N. A. On the theory of conformal mappings of a circle onto nonoverlapping regions. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 553-555. (Russian)

Let $w_k(z) = a_k + b_k z + \dots$ ($k = 1, 2$) be meromorphic and univalent in $|z| < 1$ and let the image regions D_1 and D_2 be disjoint. Let \mathcal{E} be the set of points in the (x, y) -plane ($x = |b_1|$, $y = |b_2|$) for which functions $f_k(z)$ exist satisfying the above conditions. Three theorems are stated each describing the set \mathcal{E} precisely under the following additional hypotheses: (I) D_1 and D_2 lie in the extended w -plane; (II) D_1 and D_2 lie in $|w| < \infty$; and (III) D_1 and D_2 lie in $|w| < R$.

A. W. Goodman (Lexington, Ky.).

Kővári, Tamás. On conformal mapping of ring-shaped domains. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 205-210. (Hungarian)

As a refinement of certain results of L. Bieberbach and S. E. Warschawski the author proves the following theorem. Let C be an analytic and C' any Jordan curve; we denote by ϵ their Fréchet deviation. Let z_0 be a fixed point in the interior of C . We denote by $z' = g(z)$ the function which maps the interior of C onto the interior of C' such that $g(z_0) = z_0$, $g'(z_0) > 0$. Then

$$|g(z) - z| < A \epsilon \log(1/\epsilon),$$

where A depends only on C (and on z_0). In this inequality $\log(1/\epsilon)$ can not be replaced in general by a function of $1/\epsilon$ which increases less rapidly than $\log(1/\epsilon)$. *G. Szegő*.

MacLane, Gerald R. On the Peano curves associated with some conformal maps. Proc. Amer. Math. Soc. 6 (1955), 625-630.

It is known that there exist functions $f(z)$ regular in $|z| < 1$, continuous in $|z| \leq 1$, and such that $w = f(e^{it})$ ($0 \leq t \leq 2\pi$) fills some square [see Salem and Zygmund, Duke Math. J. 12 (1945), 569-578; it is shown there that the familiar Weierstrass function $\sum a^n z^{b^n}$ has this property for suitable a and b ; MR 7, 378]. The author generalizes this result as follows. Consider a domain D in the complex plane and suppose that there is a function $f(z)$ regular in $|z| < 1$, continuous in $|z| \leq 1$, and such that the range of values of $f(z)$ in $|z| < 1$ is exactly D . Then there is a function $F(z)$ regular in $|z| < 1$, continuous in $|z| \leq 1$ and such that 1) the range of $F(z)$ in $|z| < 1$ is D , 2) the range of $F(e^{it})$ ($0 \leq t \leq 2\pi$) is the closure of D . [See also G. Piranian, C. J. Titus and G. S. Young, Michigan Math. J. 1 (1952), 69-72; MR 14, 262; A. C. Schaeffer, Duke Math. J. 21 (1954), 383-389; MR 15, 946.]

A. Zygmund (Chicago, Ill.).

Jenkins, James A. On a lemma of R. Huron. J. London Math. Soc. 30 (1955), 382-384.

The lower bound Ω for the length of curves connecting two opposite sides of a curvilinear quadrangle can be estimated from above in terms of the area σ of the quadrangle and its modulus r [Huron, Ann. Fac. Sci. Univ. Toulouse (4) 15 (1951), 155-160; MR 14, 549]. It is shown that this estimate follows easily from the method of extremal metric due to Ahlfors and Beurling. The best possible upper bound for the ratio $\Omega^2\sigma^{-1}$ in terms of r is given by use of elliptic functions. M. M. Schiffer.

Meschkowski, Herbert. Darstellung analytischer Funktionen durch den Randwinkel des Bildbereiches. Math. Z. 62 (1955), 161-166.

Let B be a simply-connected domain in the ξ -plane whose boundary curve has a piecewise continuously turning tangent. If $\xi(z)$ maps the unit circle $|z| < 1$ onto B , then $\operatorname{Re} \{1 + z\xi''/\xi'\} = d\psi/d\theta$, where $\psi(\theta)$ is the tangent angle at the point $\xi(e^{i\theta})$. Using Poisson's formula one can express $\xi'(z)$ in terms of the geometric quantity $\psi(\theta)$. In the present paper, the author uses his extension of the Poisson formula to multiply-connected domains [Ann. Acad. Sci. Fenn. Ser. A. I. no. 166 (1954); MR 15, 695] in order to give an analogous formula for such domains. A particularly simple result is obtained for the mapping of an annulus onto a doubly-connected domain. It is pointed out that in the case of higher connectivity certain linear conditions have to be imposed on the choice of $\psi(\theta)$.

M. M. Schiffer (Stanford, Calif.).

Tsuji, Masatsugu. A remark on my former paper "Theory of Fuchsian groups". J. Math. Soc. Japan 7 (1955), 202-207.

The paper referred to is in Jap. J. Math. 21, 1-27 (1952) [MR 14, 968]. For a Fuchsian group in $|z| < 1$ let $n(r, a)$ be the number of points equivalent to a in $|z| < r$ and set $N(r, a) = \int_0^r n(r, a)r^{-1}dr$. It is proved that $N(r, a) = T_0(r) + O(1)$, where $T_0(r)$ is the non-euclidean area of $|z| < r$ divided by that of a fundamental domain. It is conceded that a proof in the earlier paper was false.

L. Ahlfors (Cambridge, Mass.).

Tietz, Horst. Laurent-Trennung und zweifach unendliche Faber-Systeme. Math. Ann. 129 (1955), 431-450.

Let \mathfrak{A} be a closed Riemann surface of genus p and let C be an analytic Jordan curve that separates \mathfrak{A} into two regions G and \bar{G} . The author [Arch. Math. 4 (1953), 31-38; MR 14, 859] and H. Röhl [ibid. 3 (1952), 93-102; MR 14, 154] have presented the theory of Faber expansions of functions or differentials which are regular in $G+C$ when \bar{G} is simply-connected. In this paper the author removes the requirement that \bar{G} be simply-connected by using the mapping ψ of \bar{G} onto a region composed of a disk and N extended planes with the curve C mapping into the boundary of the disk. For functions (or differentials) which are regular in a neighborhood of C , he obtains a Laurent decomposition into the sum of two functions (or differentials), one regular in $G+C$ and the other regular in $\bar{G}+C$ except possibly for a pole at a prescribed point $b \in \bar{G}$. The functions or differentials which are regular in G (or \bar{G} except for at most a pole of order p at b) are expanded in a series of Faber functions or differentials which converges uniformly in the region of regularity. A nice symmetry is obtained by noting that the Faber functions

\mathfrak{E}_n in G and e_n in \bar{G} are just the two functions in the Laurent decomposition of ψ^{-1} , while the Faber differentials $d\mathfrak{E}_n$ in G and de_n in \bar{G} are the two differentials in the Laurent decomposition of $\psi^{-1}d\psi$. Expansions in series of Faber functions or differentials are not unique and the author discusses the question of series converging to zero. He also discusses an extension of the theory to the case in which G is bounded by r analytic curves, each homologous to zero on A . G. Springer (Lawrence, Kan.).

Rauch, H. E. On the transcendental moduli of algebraic Riemann surfaces. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 42-49.

Rauch, H. E. On moduli in conformal mapping. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 176-180.

Rauch, H. E. On the moduli of Riemann surfaces. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 236-238; errata, 421.

Through these notes the author wishes to show that it is possible to choose the requisite number of periods of normal integrals to represent the moduli of a closed Riemann surface. The point of departure is Max Noether's observation that for a non-hyperelliptic surface of genus $p \geq 3$ it is possible to find a basis for the quadratic differentials among the products $d\zeta_i d\zeta_j$ of the normal differentials. Accordingly, in the first note all considerations are limited to non-hyperelliptic surfaces.

Let S and S' be two Riemann surfaces of genus p and let the $d\zeta_i$ be normal differentials with periods π_{ij} for given canonical retrosections. We pick combinations (i, j) so that the $d\zeta_i d\zeta_j$ are a basis for the quadratic differentials on S . A topological mapping of S on S' determines corresponding retrosections, normal differentials, and periods π'_{ij} on S' . If $\pi_{ij} = \pi'_{ij}$ for the special combinations (i, j) , it is claimed that there exists a conformal mapping of S on S' which is homotopic to the given topological mapping.

To this end the author considers an extremal quasiconformal mapping of S on S' . It is determined by minimizing a mean value of the dilation rather than its maximum. In a manner that is well known from the work of Teichmüller the mapping is connected with an infinitesimal quadratic differential which will be zero if it is orthogonal to the $d\zeta_i d\zeta_j$ in a basis; its vanishing implies that the mapping is conformal.

In order to prove the required orthogonality relations the author introduces surfaces $S(t)$, intermediate between S and S' , whose fundamental form depends linearly on t , $0 \leq t \leq 1$. It is shown that one can write, symbolically, $\pi_{ij}(t) = e^{tX_{ij}}\pi_{ij}$ where the X_{ij} are infinitesimal transformations, and it is also proved that $X_{ij} = 0$ would cause the orthogonality relations to hold. Unfortunately, the indications are not sufficient to make it clear how the author passes from the assumption $\pi_{ij}(1) = \pi'_{ij}(0)$ to the conclusion $X_{ij} = 0$. Until further evidence is presented the reviewer must regard this as a very obscure link in an otherwise promising train of thought.

In the second and third note the author studies the modifications that are necessary to include bounded surfaces and the hyperelliptic case. It seems that these modifications do not introduce any essential new difficulties.

L. Ahlfors (Cambridge, Mass.).

Vekua, I. N. On a property of solutions of a generalized system of Cauchy-Riemann equations. Soobšč. Akad. Nauk Gruz. SSR 14 (1953), 449-453. (Russian)

The author is concerned with the system

$$\begin{aligned}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= a(x, y)u + b(x, y)v, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= c(x, y)u + d(x, y)v.\end{aligned}$$

In particular, using a lemma on the representation of a solution $u+iv$ of such a system as the product of two functions, one of which is an ordinary analytic function and the other of which is an exponential of a complex-valued function, the author gives a simple proof of a theorem of T. Carleman [C. R. Acad. Sci. Paris 197 (1933), 471-474] to the effect that a non-identically zero solution $u+iv$ can only have isolated zeros. [For a proof of Carleman's theorem along the same lines, cf. L. Bers, Theory of pseudo-analytic functions, Inst. Math. Mech., New York Univ., 1953, especially Note 6 on p. 171; MR 15, 211].

J. B. Diaz (College Park, Md.).

Arrighi, Gino. Sulle funzioni poldrome di matrici. Ann. Scuola Norm. Sup. Pisa (3) 8 (1954), 141-156 (1955).

The problem of defining $\varphi(A)$, where $\varphi(z)$ is a many-valued analytic function and A is a finite-dimensional matrix, has been considered by several authors in the past. The present writer returns to this topic, going over some familiar ground with due credit to his predecessors, whilst pointing out some errors and obscurities in their work and generalizing their methods. He gives a clear exposition of two ways of defining $\varphi(A)$.

The first method is a generalisation of Sylvester's interpolation formula. Let e_1, e_2, \dots, e_m be the distinct latent roots of A and let i_s+1 be the index of e_s , that is the greatest degree of an elementary divisor belonging to e_s . For each s ($1 \leq s \leq m$) choose an arbitrary branch $\varphi_s(z)$ of $\varphi(z)$ subject only to the condition that $\varphi_s(z)$ is regular at e_s . By a generalized interpolation formula one can construct a polynomial $P(z)$ of degree not greater than $m-1+(i_1+i_2+\dots+i_m)$ such that

$$\left[\left(\frac{d}{dz}\right)^j P(z)\right]_{z=e_s} = \left[\left(\frac{d}{dz}\right)^j \varphi_s(z)\right]_{z=e_s}$$

($s=1, 2, \dots, m$; $j=0, 1, 2, \dots, i_s$). The first, or "restricted", definition then consists in putting $\varphi(A)=P(A)$. This does not always yield satisfactory results; thus when $A=I$, we obtain that $\varphi(I)=\text{const} \times I$, which for $z^{1/2}$ gives only a trivial solution.

In the second method A is transformed into canonical form

$$T^{-1}AT = C \text{ diag} (\dots, C_r^{(n)}, \dots),$$

where $s=1, 2, \dots, m$ and $C_1^{(n)}, C_2^{(n)}, \dots, C_m^{(n)}$ are the submatrices belonging to the characteristic root e_s . For each pair of suffixes (s, r) one chooses an arbitrary branch of $\varphi_{s,r}(z)$ which is regular for $z=e_s$ and puts

$$\varphi^*(C) = \text{diag} (\dots, \varphi_{s,r}(C_r^{(n)}), \dots).$$

This leads to the definition

$$\varphi(A) = T\varphi^*(C)T^{-1}$$

with the understanding that T ranges over all matrices that transform A into its canonical form.

Applications are made to projective mappings and to infinite matrices which are direct sums of finite matrices.

W. Ledermann (Manchester).

Rosculeț, Marcel N. Functions of a hypercomplex variable in the n -dimensional space. Rev. Math. Phys. 2 (1954), 124-132 (1955).

The author extends his previous paper [Acad. R. P.

Române. Bul. Sti. A. 1 (1949), 523-528] to hypercomplex systems with basis $\theta, \theta^2, \dots, \theta^{n-1}$ such that $\sum_{i=0}^{n-1} \lambda_i \theta^i = 0$ where the λ_i are real. He defines norm and inverse of elements in the algebra. In part II he uses Fedorov's conditions of monogeneity [Mat. Sb. N.S. 18(60) (1947), 353-378; MR 8, 25] to determine $\int_{C_k} f(w)(w-k)^{-1}dw$. In part III he uses his norm and Fedorov's conditions of monogeneity to determine the characteristic manifolds of the algebra.

J. J. Ward (Holloman, N.M.).

See also: Clunie, p. 243; Stahl, p. 316.

Harmonic Functions, Potential Theory

Hopf, Eberhard. Die Harnacksche Ungleichung für positive harmonische Funktionen. Math. Z. 63 (1955), 156-157.

On en donne une démonstration très courte utilisant seulement le théorème de la moyenne de Gauss, et non, comme on fait habituellement, la représentation de Poisson.

J. Deny (Princeton, N.J.).

Miles, E. P., Jr., and Williams, Ernest. A basic set of homogeneous harmonic polynomials in k variables. Proc. Amer. Math. Soc. 6 (1955), 191-194.

Les auteurs donnent une formule réelle exprimant un système de polynômes homogènes, harmoniques et linéairement indépendants, de degré n en k variables ($k \geq 3$), tels que chaque polynôme harmonique de même degré et même nombre de variables en est une combinaison linéaire. En apportant de légères modifications à la formule susdite les auteurs donnent aussi des bases pour les solutions polynomiales soit de l'équation des ondes, soit des équations

$$\sum_{j=1}^k \frac{\partial^2 u}{\partial x_j^2} = 0 \quad (s=3, 4, \dots).$$

Enfin ils annoncent une note qui aurait pour but de comparer leurs résultats à ceux de M. H. Protter [Trans. Amer. Math. Soc. 63 (1948), 314-341; MR 9, 509], obtenus dans le cas $k=3$.

M. J. De Schwarz (Rome).

Cattabriga, Lamberto. Osservazioni sul problema generalizzato di Dirichlet. Rend. Sem. Mat. Univ. Padova 24 (1955), 45-52.

Let D be a finite domain in three-space whose boundary consists of finitely many simple closed surfaces admitting twice continuously differentiable parameter representations and being free of singular points. On the boundary FD of D there is prescribed a quadratically integrable function f . Let P be a boundary point and P_i the interior point at distance t on the normal at P . A harmonic function u is said to assume the boundary values f in the mean if $\lim_{t \rightarrow 0} \int_{FD} [u(P_i) - f(P)]^2 dP = 0$. The author proves that u assumes then the boundary values in the ordinary sense at every point of continuity of f . The proof uses only the most elementary properties of harmonic functions and of surfaces. It is based on an extension of the maximum principle which states that if f is almost everywhere positive and if u assumes f in the mean on FD , then u is positive in the interior of D .

W. Wasow.

Miranda, Carlo. Gli integrali principali nella teoria del potenziale. Rend. Sem. Mat. Fis. Milano 24 (1952-53), 107-122 (1954).

Exposé sur les valeurs principales d'intégrales et les

équations intégrales où figurent de telles valeurs principales (en particulier: travaux de Giraud) avec applications aux problèmes de contour pour les équations elliptiques du second ordre (en particulier: problème de la dérivée oblique).
J. Deny (Princeton, N.J.).

Mitchell, Josephine. Potential theory in the geometry of matrices. Trans. Amer. Math. Soc. 79 (1955), 401-422.

For a square matrix of complex variables $z=(z^{jk})$ ($1 \leq j, k \leq n$), denote by z^* the conjugate transposed, by D the domain $1-zz^*>0$ and by B the part of the boundary $1-zz^*=0$. It was previously proven by the reviewer [Ann. of Math. (2) 45 (1944), 686-707; MR 6, 123] that for a function $f(z)$ which is holomorphic in the variables z^{jk} in D and has continuous boundary values on B the Cauchy formula

$$f(t) = \frac{1}{V} \int_B \frac{f(z) dV}{\det^*(1-tz^*)},$$

$$dV = (-1)^{n(n+1)/4} \det^{-n} z dz^{11} dz^{12} \dots dz^{nn},$$

$$V^{-1} = \frac{1! 2! \dots (n-1)!}{(2\pi)^{n(n+1)/2}},$$

holds, and the author shows that the real part $u(z, \bar{z})$ of such a function satisfies the formula

$$u(t, \bar{t}) = \frac{1}{V} \int_B \frac{u(z, \bar{z}) \det^*(1-tz^*) dV}{\det^*(1-zt^*)(1-tt^*)}.$$

[Reviewer's remark. The author's reasoning, as it stands, presupposes that the imaginary part have likewise continuous boundary values on B , but this can be eliminated by introducing an approximating domain D_ε : $1-\varepsilon-zz^*>0$, $0<\varepsilon<1$, and letting $\varepsilon \rightarrow 0$.]

The factor of $u(z, \bar{z})$ in the author's formula is not the real part of a holomorphic function in t , but only satisfies the Laplace equation

$$\Delta_t P = 4(\delta_{ij}^2 - z^{ji} z^{ij})(\delta_{kl}^2 - \bar{z}^{lk} \bar{z}^{kl}) \frac{\partial^2 P}{\partial \bar{z}^{jk} \partial z^{il}} = 0.$$

The Laplace operator Δ_z pertains to the Riemannian metric $ds^2 = \text{trace}((1-zz^*)^{-1} dz(1-z^*z)^{-1} d\bar{z}^*)$, and both are invariant with respect to the homeomorphisms of D : $w=(az-at)(d-dt^*z)^{-1}$, t is fixed in D , $a^*a=(1-tt^*)^{-1}$, $d^*d=(1-t^*t)^{-1}$, $a^*at=td^*d$.

The author also shows that the function

$$G(z, t) = z^{-1} \log \det(1-zt^*)(z^*-t^*)^{-1}(z-t)^{-1}(1-tz^*),$$

where $t \in D$, $z \in \bar{D}$, is, locally for $z \neq t$, the real part of a holomorphic function in the variables z , and that it is the only such function which has boundary values 0 on B and whose singular part at t is a function of the quantity $v = \det^*(z-t)(z^*-t^*)$ only.
S. Bochner.

See also: Rosenbloom, p. 246.

Series, Summability

Matsuyama, Noboru. On the series with monotone terms. J. Math., Tokyo 1 (1953), 94-98.

The author considers a class of sequences generalizing Tsuchikura's generalization [Tôhoku Math. J. (2) 3 (1951), 203-207; MR 13, 737] of Szász's generalization [Amer. J. Math. 70 (1948), 203-206; MR 9, 278] of monotone sequences. This is the class of sequences $\{a_n\}$ for which

$$(*) \quad \sum_{n=1}^{\infty} b_n |\Delta a_n| + b_n a_n \leq B \sum_{n=1}^{\infty} a_n,$$

with b_n/n positive, bounded, and bounded away from 0. He proves the following results. (1) If $\{a_n\}$ satisfies the condition $0 < a_{n+1} < a_n(1+b_n^{-1})$ (which implies $(*)$) with $1 < b_n \uparrow \infty$, and if for each n there exists $m=m(n) < n$ such that $b_m/b_n \leq A$, $0 < \eta \leq (n-m)/b_m$, then $\sum a_n$ converges to s if and only if $s - K a_n b_n \rightarrow s$, where K is independent of n and $0 < K < (1-\varepsilon^{-\eta})/A$. (2) If $\{a_n\}$ is a positive sequence satisfying $(*)$, if $0 < n/b_n < \eta$ and $0 < B < \eta^{-1}(d+c)/(d-c)$, while $\varepsilon_n \geq 0$, $\limsup n^{-1} \sum_{n=1}^{\infty} \varepsilon_n = d < \infty$, $\liminf n^{-1} \sum_{n=1}^{\infty} \varepsilon_n = c$, then $\sum a_n$ and $\sum a_n \varepsilon_n$ converge or diverge together. (3) If $b_n \uparrow \infty$, $\sum 1/b_n$ diverges, $a_n > 0$, $\{a_n\}$ satisfies $(*)$, and $\sum a_n$ converges, then $a_n b_n \rightarrow 0$.
R. P. Boas, Jr.

Buck, R. C. Some remarks on Tauberian conditions.

Quart. J. Math. Oxford Ser. (2) 6 (1955), 128-131.

Ist die Folge $\{a_n\}$ von beschränktem Wachstum [d.h. zu jedem $\lambda > 1$ gibt es ein $M(\lambda)$ so, dass $0 \leq a_n \leq M(\lambda) \cdot a_n$ ist für $n \leq k \leq \lambda n$], so folgt aus $\sum a_n = s$ stets $na_n \rightarrow 0$ ($n \rightarrow \infty$). Dies verbessert ein Resultat von Szász [Amer. J. Math. 70, 203-206 (1948); MR 9, 278], wo $\{a_n\}$ quasimonoton vorausgesetzt ist [d.h. $0 \leq a_{n+1} \leq a_n(1+\alpha n^{-1})$ für ein $\alpha > 0$ und $n=1, 2, \dots$]. Der Beweis führt über folgenden Tauberiansatz: Ist $C_1\text{-}\lim a_n = 0$ und $\{a_n\}$ von beschränktem Wachstum, so gilt $\lim a_n = 0$; 0 kann durch keine andere Zahl ersetzt werden. Ferner wird bewiesen: Ist in der Folge $\{a_n\}$ die Teilfolge $\{a_{r_n}\}$ konvergent gegen Null, wobei $r_{n+1}/r_n = O(1)$ (bzw. $r_{n+1}/r_n \rightarrow 1$) ist, und ist $\{a_n\}$ von beschränktem Wachstum (bzw. langsam oszillierend), so gilt $\lim a_n = 0$. Eine Anwendung auf ganze Funktionen wird gegeben. Wegen einer Verallgemeinerung des Szászschen Satzes in anderer Richtung vgl. Matsuyama [oben referierte Arbeit].
D. Gaier (Stuttgart).

Zaring, W. M. Multiply monotone complex sequences.

Proc. Amer. Math. Soc. 6 (1955), 583-593.

D'après K. Knopp [Math. Z. 22 (1925), 75-85], une suite de nombres réels $\{a_n\}$ est monotone d'ordre $\alpha \geq 0$ si

$$\Delta^\alpha a_n \geq 0, \text{ où } \Delta^\alpha a_n = \sum_{k=0}^{\infty} \binom{k-\alpha-1}{k} a_{n+k}, (n=1, 2, \dots).$$

L'auteur introduit l'extension suivante: Une suite de nombres complexes $\{c_n\} = \{a_n + ib_n\}$ est (α, ϕ) -monotone, si $|\arg \Delta^\alpha c_n| \leq \phi < \pi/2$, ou, ce qui revient au même, si $|\Delta^\alpha b_n| \leq \tan \phi \Delta^\alpha a_n$ ($n=1, 2, \dots$). Alors, il déduit parmi les autres, de convergence de $\sum c_n$ et du fait que $\{c_n\}$ est (α, ϕ) -monotone ($0 \leq \alpha \leq 1$) résulte $n^\alpha c_n \rightarrow 0$. L'auteur donne ensuite plusieurs extensions des théorèmes de Fejér [Trans. Amer. Math. Soc. 39 (1936), 18-59]. Le résultat suivant est typique. Si $\{c_n\}$ est (α, ϕ) -monotone, $c_n \rightarrow 0$, $f(\theta) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos n\theta$, $g(\theta) = \sum_{n=1}^{\infty} c_n \sin n\theta$, alors $f(\theta)$ et $g(\theta)$ convergent, $|\arg f(\theta)| \leq \phi$ ($0 < \theta < 2\pi$), $|\arg g(\theta)| \leq \phi$ ($0 < \theta < \pi$).
M. Tomić (Beograd).

Wheelon, Albert D. Note on the summation of finite series. J. Math. Phys. 34 (1955), 182-186.

Examples are given which illustrate the familiar procedure of introducing suitable integral representations for the terms of a sum to obtain a useful formula giving the sum as the integral of the product of a function and a simpler sum. The examples involve factorials and Laplace transforms.
R. P. Agnew (Ithaca, N.Y.).

Teghem, J. Remarques sur les transformations de Taylor et de Laurent. Acad. Roy. Belg. Bull. Cl. Sci. (5) 41 (1955), 719-722.

The author points out certain connections between a paper of his [same Bull. (5) 36 (1950), 730-741; MR 12,

695] and a recent paper by Meyer-König and Zeller [Math. Z. 60 (1954), 348-352; MR 16, 28]. He also discusses the consistency of Laurent summability as applied to power series in the region of regularity. R. C. Buck.

Štegllov, M. P. On two theorems of Hardy-Littlewood. Ukrain. Mat. Ž. 7 (1955), 180-187. (Russian)

Let $a_0 + a_1 + a_2 + \dots$ be a series with real terms and partial sums $s_n = a_0 + a_1 + \dots + a_n$. Let $A(u) = \sum_{k=0}^{\infty} a_k e^{-ku}$ denote the Abel transform of $\sum a_n$. Let

$$d = \liminf_{n \rightarrow \infty} s_n, \quad d' = \liminf_{u \rightarrow 0} A(u), \\ D' = \limsup_{n \rightarrow \infty} A(u), \quad D = \limsup_{u \rightarrow 0} s_n.$$

It is a well known fact that these four numbers satisfy the inequality $d \leq d' \leq D' \leq D$. While the reviewer does not recall seeing the fact stated in print, it is possible to construct a series for which d, d', D', D are any pre-assigned numbers, finite or infinite, that satisfy this inequality. If, however, T is a Tauberian condition such that each series satisfying T and evaluable A is convergent, then we can assert that if $\sum a_n$ is a series satisfying T then the relations $d < d' = D'$ and $d' = D' < D$ are impossible because if $d' = D'$, then $d = d' = D' = D$. In a previous paper the author [Mat. Sb. N.S. 28(70) (1951), 245-282; MR 13, 28] made an exhaustive study of the relations among $d, d', D',$ and D that are possible when $\sum a_n$ satisfies an order Tauberian condition $na_n = o(1)$ or $na_n = O(1)$ or $na_n < o(1)$ or $na_n < O(1)$. The present paper treats the same problem with $\sum a_n$ satisfying a Tauberian gap condition, that is, $a_n = 0$ except when $n = n_1, n_2, n_3, \dots$, where n_k is a rapidly increasing sequence of integers. R. P. Agnew (Ithaca, N.Y.).

Čakalov, Lyubomir. Generalization of a convergence theorem of Mercer. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. 1 (1954), no. 2, 85-89. (Bulgarian. Russian summary)

In standard notation and terminology, a sequence $\lambda_0, \lambda_1, \dots$ for which $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \dots$ and $\lambda_n \rightarrow \infty$ determines a Riesz transformation

$$\sigma_n = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) u_k$$

by means of which a given series $\sum u_k$ is evaluable to L if $\sigma_n \rightarrow L$ as $n \rightarrow \infty$. In terms of the partial sums $x_n = u_0 + \dots + u_n$ of $\sum u_k$, this becomes

$$\sigma_n = \frac{1}{\lambda_{n+1}} \sum_{k=0}^n (\lambda_{k+1} - \lambda_k) x_k$$

and in case $\lambda_n = n$ it reduces to the arithmetic mean transformation C_1 . The classic theorem of Mercer applies to C_1 only. The paper under review extends the theorem of Mercer to the whole class of these Riesz transformations (or order 1) by proving that if λ is a complex number whose real part exceeds -1 , and if the sequence $\lambda \sigma_n + x_n$ is convergent, then the sequence x_n and the series $\sum u_k$ must be convergent.

Actually the above formulation of the results is a little different from that of the author. Suppose that, with the above notation, $\lim (\lambda \sigma_n + x_n)$ exists. Then clearly

$$\lim_{n \rightarrow \infty} \left\{ \frac{\lambda}{\lambda_{n+1}} [\lambda_1 x_1 + \sum_{k=1}^n (\lambda_{k+1} - \lambda_k) x_k] + x_n \right\}$$

exists. On setting $p_1 = \lambda_1$, $p_k = \lambda_{k+1} - \lambda_k$ when $k > 1$, and $P_n = p_1 + \dots + p_n = \lambda_{n+1}$, and defining y_n by

$$y_n = (p_1 x_1 + p_2 x_2 + \dots + p_n x_n) / P_n,$$

we see that $0 < p_1 < p_2 < \dots$, $P_n \rightarrow \infty$, and $\lim (\lambda y_n + x_n)$ exists. These are the hypotheses used by the author to show that $\lim x_n$ exists. In the proof, p_{k-1} is misprinted for P_{k-1} in a definition following the author's equation (6). R. P. Agnew (Ithaca, N.Y.).

Ramanujan, M. S. On Hausdorff transformations for double sequences. Proc. Indian Acad. Sci. Sect. A. 42 (1955), 131-135.

Conditions under which one factorable summability scheme for double sequences includes another are known [Adams, Trans. Amer. Math. Soc. 34 (1932), 215-230] as are conditions guaranteeing the regularity of the Hausdorff summability method (H, μ) [Adams, Bull. Amer. Math. Soc. 39 (1933), 303-312]. Using these and assuming that $\mu_{mn} \neq 0$ for all m and n , the author shows that for the class of double sequences bounded (H, μ) (bounded (H, ν) if μ and ν are factorable), $(H, \mu)C(H, \nu)$ if and only if ν_{mn}/μ_{mn} is a regular moment sequence. He also gives a "Mercer theorem" [Hardy, Divergent series, Oxford, 1949, pp. 104, 264; MR 11, 25] for double sequences. A. E. Livingston (Seattle, Wash.).

Watanabe, Hiroshi. On some summations of double series. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. 9 (1955), 47-54.

Several facts, some of which are known, about Cesàro and Abel evaluability of double series and Cauchy products of simple series are obtained as applications of the following theorem. In order that a matrix a_{jk} of real [or complex] constants be such that

$$\lim_{n \rightarrow \infty} \sum_{j+k=n} a_{jk} s_{jk} = s$$

whenever s_{jk} is a bounded sequence that converges to s , it is necessary and sufficient that (i) for each $j=0, 1, 2, \dots$, $a_{jk} \rightarrow 0$ as $k \rightarrow \infty$ and for each $k=0, 1, 2, \dots$, $a_{jk} \rightarrow 0$ as $j \rightarrow \infty$; (ii) there exist a constant M such that $\sum_{j+k=n} |a_{jk}| < M$ for each n ; and (iii) $\sum_{j+k=n} a_{jk} \rightarrow 1$ as $n \rightarrow \infty$. R. P. Agnew (Ithaca, N.Y.).

Rajagopal, C. T. A note on Ingham summability and summability by Lambert series. Proc. Indian Acad. Sci. Sect. A. 42 (1955), 41-50.

Let $0 < \lambda_1 < \lambda_2 < \dots$ and $\lambda_n \rightarrow \infty$. A series $a_1 + a_2 + \dots$ is evaluable to s by the Lambert method L_λ of type λ if the series in

$$L_\lambda(t) = \sum_{n=1}^{\infty} \frac{\lambda_n t}{e^{\lambda_n t} - 1} a_n$$

converges when $t > 0$ and $L_\lambda(t) \rightarrow s$ as $t \rightarrow 0$. Let k be a constant not necessarily an integer. The series $\sum a_n$ is evaluable to s by the Ingham method $I_{\lambda,k}$ of type λ and order k if the transform defined by

$$I_{\lambda,k}(x) = (k+1) \sum_{\lambda_n < x} a_n \frac{\lambda_n}{x} \sum_{i < n/\lambda_n} \left(1 - \frac{i \lambda_n}{x}\right)^k$$

is such that $I_{\lambda,k}(x) \rightarrow s$ as $x \rightarrow \infty$. In case $\lambda_n = n$ and $k=0$, $I_{\lambda,k}(x)$ reduces to $\sum_{n \leq x} nx^{-1} [xn^{-1}] a_n$ and thus $I_{\lambda,k}$ reduces to the Ingham method I treated in the textbook of Hardy [Divergent series, Oxford, 1949; MR 11, 25]. W. B. Pennington [Proc. Cambridge Philos. Soc. 51 (1955), 65-80; MR 16, 465] has shown that the methods $I_{\lambda,k}$ are related to each other and to L_1 in much the same way that the Riesz typical methods $R_{\lambda,k}$ are related to each other and to the Abel method A_λ . The main part of the paper under review supplements results of Pennington. The following and related facts are obtained. Let $\sum a_n$

be a real series for which the Lambert transform $L_A(t)$ exists. Then

$$\limsup_{t \rightarrow \infty} L_A(t) = \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} I_{A,k}(x)$$

provided

$$\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} I_{A,k}(x) > -\infty.$$

Connections between $L_A(t)$ and other transforms of the form $\sum a_n \phi(\lambda_n t)$ are discussed. An appendix of the paper gives an elementary but rather complicated example, constructed by T. Vijayaraghavan, of a convergent series nonevaluable $I_{A,k}$ when $\lambda_n = n$ and $k=0$. That $I_{A,k}$ is, in this case, not regular was stated by Hardy [loc. cit.] and was proved by Pennington [loc. cit.] by a method requiring use of nontrivial information from analytic number theory. R. P. Agnew (Ithaca, N.Y.).

Rajagopal, C. T. A generalization of Tauber's theorem and some Tauberian constants. III. Comment. Math. Helv. 30 (1956), 63-72 (1955).

[For parts I and II see Math. Z. 57 (1953), 405-414; 60 (1954), 142-147; MR 14, 958, 16, 124.] The following Tauberian theorem and several of its consequences are obtained. Let $A(u)$ be a complex-valued function having bounded variation over each finite interval $0 \leq u \leq u_0$ and let $A(u)$ satisfy the Tauberian condition

$$\limsup_{u \rightarrow \infty} |A(u) - u^{-1} \int_0^u A(x) dx| \leq Q < \infty.$$

Let $N(x)$ be a complex-valued function which is bounded over each finite interval $0 \leq x < x_0$ and let $N(x)$ and $N(x) \log x$ be Lebesgue integrable over $0 < x < \infty$. Let $\int_0^\infty N(x) dx = 1$ and let $K(u) = \int_0^\infty N(x) dx$. Then, for each $\delta > 0$,

$$(1) \quad \limsup_{t \rightarrow 0+} |A(\delta/t) - \int_0^\infty K(u) dA(u)| \leq T^*(\delta) Q,$$

where $T^*(\delta)$ is the constant independent of $A(u)$ defined by

$$T^*(\delta) = 1 + \int_0^\delta \left| \frac{1-K(x)}{x} - N(x) \right| dx + \int_\delta^\infty \left| \frac{K(x)}{x} + N(x) \right| dx.$$

Moreover, it is efficiently shown by use of an appropriate lemma that (1) is an optimal inequality in the sense that the result of replacing \leq by $<$ is invalid. R. P. Agnew.

Korevaar, Jacob. Tauberian theorems. Simon Stevin 30 (1955), 129-139.

This is an expository lecture in which the author reaches the conclusion that a method of Wielandt [Math. Z. 56 (1952), 206-207; MR 14, 265] provides an improved way of obtaining results such as those which the author [Duke Math. J. 18 (1951), 723-734; MR 13, 227] has obtained by other methods. A complete account of his findings will be published elsewhere. R. P. Agnew.

Chen, Kien-Kwong. Ikehara's theorem and absolute summability C. Acta Math. Sinica 3 (1953), 8-11. (Chinese. English summary)

In a study of a theorem of Ikehara, Wintner [Amer. J. Math. 69 (1947), 99-103; MR 8, 375] constructed a series $a_1 + a_2 + \dots$ for which (i) the Dirichlet series $\sum a_n n^{-(s+it)}$ converges absolutely when $\sigma > 1$; (ii) the power series $\sum a_n x^n$ converges over $0 < x < 1$ to a function $f(x)$ for which $\int_0^1 |f(x)| dx < \infty$; and (iii) the series $\sum a_n$ is not evaluable C_1 to 0, i.e. $n^{-1} \sum_{k=1}^n a_k \neq o(1)$. Using the fact that there exists a divergent series $\sum c_n$ absolutely evaluable C_α for each $\alpha > 0$, the author constructs a series $\sum a_n$ for which

(i) and (iii) hold and, in addition to (ii), the function $f(x)$ in (ii) has bounded variation over $0 < x < 1$.

R. P. Agnew (Ithaca, N.Y.).

Djerasimović, Božidar. Beitrag zur Theorie der regelmässigen Kettenbrüche. Math. Z. 62 (1955), 320-329.

The author introduces a set of symbols to express certain operations on ordered n tuples of numbers. For example: if $A = a_1, \dots, a_n$ then $\bar{A} = a_n, \dots, a_1$; $A' = a_2, \dots, a_n, A' = a_1, \dots, a_{n-1}$. The purpose of this notation is to facilitate the derivation of results in the theory of regular continued fractions. New proofs for classical results as well as for some recent theorems due to Steinerwald [Math. Z. 52 (1950), 686-697; MR 12, 393] are then given using this new symbolism. W. J. Thron.

See also: Allen, p. 283.

Interpolation, Approximation, Orthogonal Functions

Natucci, A. Il problema dell'interpolazione. Giorn. Mat. Battaglini (5) 3(83) (1955), 89-106.
Report of a lecture given in 1949.

Davis, Philip. On a problem in the theory of mechanical quadratures. Pacific. J. Math. 5 (1955), 669-674.
Convergence to zero of

$$Q_n(f) = \int_{-1}^{+1} f(x) dx - \sum_{k=0}^n a_{nk} / (\lambda_{nk})$$

is discussed for functions $f(z)$ analytic in the ellipse E_ρ with foci $-1, +1$ and the sum of semi-axes $\rho > 1$. The main result is as follows. If $\rho > 4$ and the approximating sum is the Newton-Cotes quadrature, then $\|Q_n\| \rightarrow 0$, if $\|Q_n\|$ is the norm of $Q_n(f)$ as a linear functional on the Hilbert space of functions $f(z)$ analytic in E_ρ with $\iint |f(z)|^2 dx dy < +\infty$. An estimate of Newton-Cotes coefficients due to Ouspensky [Bull. Amer. Math. Soc. 31 (1925), 145-156] is used. G. G. Lorentz.

Fuchs, W. H. J., and Pollard, Harry. A note on Bernstein's approximation problem. Proc. Amer. Math. Soc. 6 (1955), 613-615.

Vidav [C.R. Acad. Sci. Paris 238 (1954), 1959-1961; MR 15, 870] avait montré que certaines conditions de Pollard (existence d'une suite de polynômes p_n tels que $p_n(u)K(u) \rightarrow 1$ et $\sup_{n,u} |p_n(u)K(u)| < \infty$) jointes à l'hypothèse (V): K n'est pas l'inverse d'une fonction entière, entraînent que K est une fonction-poids de Bernstein. Les auteurs remplacent (V) par (V'): K n'est pas l'inverse d'une fonction entière de type exponentiel nul.

J. P. Kahane (Montpellier).

Wermer, John. Polynomial approximation on an arc in C^2 . Ann of Math. (2) 62 (1955), 269-270.

Soit γ un arc d'aire positive dans le plan, et A la classe des fonctions analytiques dans le complémentaire C_γ de γ (∞ compris) et continues dans tout le plan; A contient d'autres fonctions que les constantes d'après Denjoy [Bull. Soc. Math. France 60 (1932), 27-105]; visiblement, A est fermée pour la convergence uniforme sur γ . L'auteur montre 1) que pour toute $f \in A$, on a $f(\gamma) \supset f(C_\gamma)$, 2) qu'il existe 3 fonctions F_1, F_2, F_3 dans A séparant les points de γ . De 1) résulte l'existence (déjà

connue) d'une fonction analytique sur $|z| < 1$, continue sur $|z| \leq 1$ et prenant à la frontière toute valeur prise à l'intérieur. De 1) et 2) résulte l'existence dans C^2 d'un arc Γ (image de γ par $\{F_1, F_2, F_3\}$) tel que toute fonction $f(z_1, z_2, z_3)$ à valeurs complexes non constante, uniformément approchable sur Γ par des polynômes $P(z_1, z_2, z_3)$, applique Γ sur une courbe de Peano. Enfin on remarque que, restreinte à γ , A constitue une sous-algèbre propre de l'algèbre $C(\gamma)$ des fonctions à valeurs complexes continues sur γ , contenant l'unité, fermée, séparante (d'après 2)) et contenant f^{-1} si $f \in A$ ne s'annule pas (d'après 1)); d'ici l'existence d'une telle sous-algèbre dans $C(0, 1)$.

L. P. Kahane (Montpellier).

APARICIO BERNARDO, E.

Aparicio Bernardo, E. On some properties of polynomials with integral coefficients and on approximation of functions in the mean by polynomials with integral coefficients. *Izv. Akad. Nauk SSSR. Ser. Mat.* 19 (1955), 303-318. (Russian)

The problem of approximation of a function on an interval I of length λ by polynomials with integral coefficients in the L^2 or the C norm is closely connected with the existence of nonzero polynomials with integral coefficients and arbitrary small norm on I . The existence of such P_n for the L^2 metric and $\lambda < 4$ was proved by Hilbert [*Acta Math.* 18 (1894), 155-159], for the C metric by Fekete [*Math. Z.* 17 (1923), 228-249]. For $\lambda \geq 4$, the P_n do not longer exist. The author proves the following. (1) Let $E_n(f)$ and $E_n^0(f)$ be the best quadratic approximations of a function $f(x)$ on $(0, \lambda)$ by ordinary polynomials and by polynomials with integral coefficients, of degree n . If $P(x)$ is a polynomial with integral coefficients such that $0 \leq P(x) < 1$ on $(0, \lambda)$ and μ is the highest multiplicity of its zeros on $(0, \lambda)$, then $E_n^0(f) \leq 2E_n(f) + O(n^{-1/\mu})$ provided $\lambda < 4$; if $\mu = 0$, the last term is to be replaced by q^n with some $0 < q < 1$. In particular, $E_n^0(f) \rightarrow 0$ for $f \in L^2$ and $\lambda < 4$. (2) If $0 \leq \lambda_1 < \lambda_2 < \dots$, $S_n = \sum_{k=1}^n \lambda_k$ and $S_n/\lambda_n^2 \rightarrow \infty$ then for any $\lambda > 0$, $\varepsilon > 0$ there is a sum $P(x) = \sum_{k=1}^n a_k x^{\lambda_k}$ with integral coefficients such that $0 < \int_0^\lambda P(x)^2 dx < \varepsilon$. The proofs depend on Minkowski's theorem on linear forms and on an idea of Kantorovič who proved a result corresponding to (1) for $I = (0, 1)$ and uniform approximation [Lorentz, Bernstein polynomials, Univ. of Toronto Press, 1953, pp. 49-50; MR 15, 217]. G. G. Lorentz.

Bari, N. K. On best approximation of two conjugate functions by trigonometric polynomials. *Izv. Akad. Nauk SSSR. Ser. Mat.* 19 (1955), 285-302. (Russian)

Given a continuous $f(x)$ of period 2π , we denote by $E_n(f)$ the best approximation of f by trigonometric polynomials of order n or less. If

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

we may compare $E_n(f)$ and $E_n(f)$, provided

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$$

is continuous. We say that $\varphi(\delta)$ ($0 \leq \delta \leq 2\pi$), belongs to class Φ , if φ is non-decreasing, zero at $\delta = 0$ but nowhere else. The main result of the paper is as follows. Suppose that $\varphi \in \Phi$, and consider all continuous f of period 2π such that (*) $E_n(f) = O(1/n)$. Then, if

$$(\dagger) \quad \sum_{k=n}^{\infty} k^{-1} \varphi(k^{-1}) = O(\varphi(n^{-1})),$$

every f satisfying (*) satisfies also (**) $E_n(f) = O(\varphi(n^{-1}))$.

If $\sum k^{-1} \varphi(k^{-1})$ converges, but (\dagger) does not hold, we can find a continuous f satisfying (*) and such that f is continuous but does not satisfy (**). Finally, if $\sum k^{-1} \varphi(k^{-1})$ diverges, there is a continuous f satisfying (*) with f discontinuous (unbounded). Furthermore, it is shown that if $\varphi \in \Phi$, condition (\dagger) is equivalent to the existence of a constant $C > 1$ such that

$$\liminf_{\delta \rightarrow 0} \varphi(C\delta)/\varphi(\delta) > 1.$$

A. Zygmund (Chicago, Ill.).

Harrik, I. Yu. On approximation of functions vanishing on the boundary of a region by functions of a special form. *Mat. Sb. N.S.* 37(79) (1955), 353-384. (Russian)

For many applications, e.g., below, the following theorem in estimation is derived: Let D be a bounded closed region in m -dimensional space, φ a function defined in D and such that (a) $\varphi(P) \neq 0$ if P is in the interior of D , $\varphi(P) = 0$ if P is on the boundary Γ of D . (b) φ has partial derivatives of order $k \geq 1$ and its derivatives of order k are Lip 1. (c) $|\text{grad } \varphi(P)| > 0$ if $P \in \Gamma$. Then if u is continuously differentiable k times in D and is 0 on Γ , there is a sequence of polynomials $P_n(x_1, \dots, x_m)$ of degree $\leq n$ in each variable x_i such that

$$\|u - \varphi P_n\|_{C^r(D)} = O((\omega_{C(D)}^{(k)}(u; 1/n))/n^{k-r}) \text{ for } r = 0, 1, 2, \dots, k.$$

Here

$$\|f(P)\|_{C^r(D)} = \max_{0 \leq k_1 + \dots + k_m \leq k} \max_D \left| \frac{\partial^{k_1 + \dots + k_m} f(P)}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} \right|$$

and

$$(\omega_{C(D)}^{(k)}(f; \varepsilon)) = \max_{0 \leq k_1 + \dots + k_m \leq k} \max_{\substack{P, Q \in D \\ \varrho(P, Q) \leq \varepsilon}} \left| \frac{\partial^{k_1 + \dots + k_m} [f(P) - f(Q)]}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} \right|.$$

One application given is to the classical eigenvalue problem: $\Delta u + \lambda u = 0$ in D , $u = 0$ on Γ , as solved in its variational form by the Ritz method. Let $\varphi = 0$ be the equation of Γ and P_n the polynomials of the theorem quoted. Then for $\tilde{u}_n = \varphi P_n / \|\varphi P_n\|_{L_2}$, the following estimate is valid for $u^{(1)}$, the first eigenfunction,

$$\|\tilde{u}_n - u^{(1)}\|_{C(D)} = O((\omega_{C(D)}^{(k)}(u^{(1)}; 1/n))/n^{k-1}) = O(A_n)$$

and for the corresponding approximants λ_n to the first eigenvalue $\lambda^{(1)}$ we have

$$\lambda_n - \lambda^{(1)} = O(A_n^2).$$

Details are too fine for inclusion in this review. A basic tool is the set of estimates associated with the operators T_n , $F_n = I - T_n$ and $\Psi_{\varphi, n}(f) = \varphi F_n f - F_n(\varphi f)$, where $T_n(f)$

$$= \int_{-\infty}^{\infty} f(\xi_1 + \lambda_1, \dots, \xi_m + \lambda_m) K_n(\lambda_1) \dots K_n(\lambda_m) d\lambda_1 \dots d\lambda_m$$

and K_n is Jackson's kernel:

$$\left\{ \int_{-\infty}^{\infty} \left[\frac{\sin \frac{1}{2} r \lambda}{\sin \frac{1}{4} \lambda} \right]^4 d\lambda \right\}^{-1} \left[\frac{\sin \frac{1}{2} r \lambda}{\sin \frac{1}{4} \lambda} \right]^4, \quad r = [\frac{1}{2}n] + 1.$$

B. R. Gelbaum (Minneapolis, Minn.).

Geronimus, Ya. L. On some local properties of orthogonal polynomials. *Dokl. Akad. Nauk SSSR (N.S.)* 103 (1955), 185-188. (Russian)

In an earlier note [same Dokl. (N.S.) 83 (1952), 5-8; MR 13, 740] the author gave sufficient conditions for the boundedness of a system of orthogonal polynomials over the whole interval J of orthogonality. Now he considers

local boundedness, i.e., at a point or subinterval of J . The proofs of his theorems involve notation from the earlier note, which for reasons of brevity were not mentioned in the review of that note, and can therefore not be indicated here. Sample results follow.

I. Let the polynomials $\{\hat{P}_n(z)\}$ be orthonormal on the unit circle relative to $d\sigma(\varphi)$, where i) $\log \sigma(\varphi) \in L$; ii) on the interval $I \subset [0, 2\pi]$, $\sigma(\varphi)$ is absolutely continuous, $\hat{P}(\varphi) = \sigma'(\varphi) \geq m > 0$, and $\hat{P}(\varphi) \in \text{Lip } \alpha$ ($\frac{1}{2} < \alpha \leq 1$). Then for each point $\varphi_0 \in I_0$ (I_0 lying interior to I), $|\hat{P}_n(e^{i\varphi_0})| \leq A$ (independent of n). II. Let $\hat{P}_n^*(z) = z^n \hat{P}_n(1/z)$. Under the hypotheses of I, $(*) \lim_{n \rightarrow \infty} \hat{P}_n^*(e^{i\varphi_0}) = \pi(e^{i\varphi_0})$, $\varphi_0 \in I_0$, where $\pi(e^{i\varphi_0})$ is the radial limit of $\pi(re^{i\varphi_0})$ with

$$\pi(z) = \exp \left\{ \frac{-1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \hat{P}(\theta) d\theta \right\} \quad (|z| < 1).$$

Relation (*) was obtained by Szegő under less general assumptions [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; MR 1, 14]. III. Let $\{\varphi_n(x)\}$, orthonormal relative to $d\psi(x)$ on $[-1, 1]$, be bounded at x_0 , and let the interval I contain x_0 in its interior. Suppose $d\psi(x) = w(x)dx$, $w(x) \leq C$ on I . Let $f(x)$ be such that $\int_{-1}^1 |f(x)|^2 d\psi(x) < \infty$, and satisfy on I a Lipschitz condition of order α ($\frac{1}{2} < \alpha \leq 1$). Then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n c_k \hat{\varphi}_k(x_0) = f(x_0), \text{ where } c_k = \int_{-1}^1 f(x) \hat{\varphi}_k(x) d\psi(x).$$

I. M. Sheffer (University Park, Pa.).

Geronimus, Ya. L. On properties of some orthogonal series. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 353-356. (Russian)

The author considers polynomials $P_n(z)$ that are orthogonal on the unit circumference with weight $d\sigma(\phi)$, where $\sigma(\phi)$ is bounded and nondecreasing over $(0, 2\pi)$, and satisfies $\int_0^{2\pi} \log \sigma'(\phi) d\phi > -\infty$. For series $f(z) = \sum_{k=0}^{\infty} b_k P_k(z)$ he establishes analogues of familiar power series theorems. (1) If $\sigma'(\phi) \geq m > 0$ almost everywhere, then $b_n = o(n^{-1/2})$ is sufficient for the equivalence of the two conditions (i) $\lim_{r \rightarrow 1} f(re^{i\phi}) = f(e^{i\phi})$ at almost every point where $\sigma'(\phi) > 0$, and (ii) $\sum b_n P_n(e^{i\phi}) = f(e^{i\phi})$. (2) If $\sigma(\phi)$ is absolutely continuous and $0 < m \leq \sigma'(\phi) \leq M$ almost everywhere, then (i) and (ii) are equivalent under various conditions on the integral modulus of continuity $\omega_2(\sigma'; \delta)$ together with order conditions on b_n . (3) If $\sigma(\phi)$ is absolutely continuous, let P_n be uniformly bounded on a subinterval I of $[0, 2\pi]$. If $f(\phi)$ has radial boundary values on I , belonging to Lip 1, then (ii) holds on any interval strictly interior to I .

R. P. Boas, Jr. (Evanston, Ill.).

Freud, G. Über das gliedweise Differenzieren einer orthogonalen Polynomreihe. Acta Math. Acad. Sci. Hungar. 6 (1955), 221-226. (Russian summary)

Let $\hat{P}_n(x)$ be the orthonormal polynomials of degree n with respect to the non-negative weight function $w(x) \in L$ in $(-1, 1)$. The following theorem is proved: Suppose that $w(x) \geq m > 0$ in $\langle a, b \rangle$ where $-1 < a < b < 1$. Let $f(x)$ have k continuous derivatives in $(-1, 1)$, and let $\omega(\delta, f^{(k)})$ be the modulus of continuity for $f^{(k)}$. If $\sum_{n=1}^{\infty} n^{-k} \omega(1/n, f^{(k)}) < \infty$, then the k -times differentiated Fourier series of f (with respect to the \hat{P}_n) is absolutely convergent, uniformly, in every closed subinterval of $\langle a, b \rangle$, and its sum is $f^{(k)}(x)$. A second theorem of a similar kind is also proved.

W. W. Rogosinski (Newcastle-upon-Tyne).

Alexits, G. Über die Konvergenz der Orthogonalpolynomentwicklungen. Acta Math. Acad. Sci. Hungar. 6 (1955), 1-4. (Russian summary)

Let $w(x) \geq 0$ be a weight function, L -integrable over $(-1, 1)$, and let $\hat{P}_n(x)$ denote the corresponding orthonormal polynomial of degree n . The following theorem is proved: Suppose that $0 < w(x) = O((1-x^2)^{-1/2})$ and suppose that the $\hat{P}_n(x)$ are uniformly bounded in some interval $\langle a, b \rangle$ where $-1 < a < b < 1$. Let $f(x) \in L^2$ (with respect to w), and let

$$\omega(\delta, f; a, b) = \sup_{\substack{|\lambda| \leq \delta \\ a \leq x \leq b}} |f(x+\lambda) - f(x)| = O(1/\sqrt{\lambda(1/\delta)}),$$

where $\lambda(x)$ increases and $\int_{-1}^1 x \lambda(x)^{-1} dx < \infty$. Then the Fourier series of $f(x)$, with respect to the $\hat{P}_n(x)$, converges p.p. in $\langle a, b \rangle$. The proof is by reduction to a former result of the author for trigonometrical Fourier series [same Acta 4 (1953), 95-101; MR 15, 619].

W. W. Rogosinski (Newcastle-upon-Tyne).

Yano, Shigeki. A convergence test for Walsh-Fourier series. Tôhoku Math. J. (2) 6 (1954), 226-230.

The author proves that the convergence test of Hardy and Littlewood is valid for Walsh-Fourier series, that is, the pair of conditions

$$(i) \int_0^h |f(x_0+t) - f(x_0)| dt = o(|h|/\log |h|^{-1}), \quad h \rightarrow 0, \\ (ii) \quad c_n = O(n^{-\delta}) \quad (\delta > 0),$$

where c_n is the n th Walsh-Fourier coefficient of f , is sufficient for convergence of the Walsh-Fourier series of f to $f(x_0)$ at the point x_0 . In the course of the proof, he demonstrates the Parseval equality $\int_0^1 f(x)h(x)dx = \sum a_n b_n$ for f integrable and h of bounded variation, where a_n, b_n are the Walsh-Fourier coefficients of f and h .

N. J. Fine (Philadelphia, Pa.).

Hirschman, I. I. The decomposition of Walsh and Fourier series. Mem. Amer. Math. Soc. no. 15 (1955), 65 pp.

The author generalizes inequalities for Fourier series and Walsh expansions of Paley, Littlewood, Riesz, Marcinkiewicz, Babenko and others by introducing a weighting factor in certain means.

The Walsh functions $\psi(k, t)$ ($k=0, 1, \dots; 0 \leq t \leq 1$) are defined by $\psi(k, t) = \phi(n_1, t)\phi(n_2, t) \dots \phi(n_r, t)$, where

$$\phi(n, t) = \text{sign} \sin(2^{n+1}\pi t)$$

and $k=2^{n_1}+2^{n_2}+\dots+2^{n_r}$ is the dyadic representation of n . If $f(t) \in L(0, 1)$, $c(o) = \int_0^1 f(t)dt = 0$, we have

$$f(t) \sim \sum_{k=0}^{\infty} c(k) \psi(k, t), \quad c(k) = \int_0^1 f(t) \psi(k, t) dt,$$

with the formal decompositions

$$f(t) = \sum_{n=0}^{\infty} f_n(t), \quad f_n(t) = \sum_{k=2^n}^{2^{n+1}-1} c(k) \psi(k, t), \quad c(k) = \sum_{n=1}^{\infty} c_n(k),$$

$$c_n(k) = \int_{2^{-n}}^{2^{-n+1}} f(t) \psi(k, t) dt.$$

Then if $w(\varphi) = w_{\alpha, \beta}(\varphi) = \left(\int_0^1 t^{2\alpha} |\varphi(t)|^2 dt \right)^{1/2}$ and $w(a) = w_{\alpha, \beta}(a) = \left(\sum_{k=0}^{\infty} (k+1)^{2\alpha} |a(k)|^2 \right)^{1/2}$ for functions $\varphi(t)$ in $(0, 1)$ or sequences $a(k)$, respectively, it is shown that

$$(1) \quad A' \leq w(F)/w(f) \leq A'', \quad A' \leq w(C)/w(c) \leq A''$$

when $F(t)^2 = \sum_n |f_n(t)|^2$, $C(k)^2 = \sum_n |c_n(k)|^2$ and $1 < \beta < \infty$, $-1/\beta < \alpha < 1-1/\beta$. Further, if $S_m = S_m(t) = \sum_{k=0}^m c(k) \psi(k, t)$, $S_n = S_n(k) = \sum_{k=0}^n f(k) \psi(k, t) dt$, then $w(S_m) \leq A w(f)$ and $w(S_n) \leq A w(c)$. The case $\alpha=1$ of the first inequality (1) is due to

Paley [Proc. London Math. Soc. (2) 34 (1932), 241-264].

The results for Fourier series follow a similar pattern. If $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ ($-\pi \leq \theta < \pi$), we have the decomposition

$$f(\theta) \sim \sum_n \Delta_n(\theta), \Delta_n(\theta) = \sum_{k=1}^{2n-1} a_k e^{ik\theta} \text{ or } \sum_{k=-2n+1}^{-n+1} a_k e^{ik\theta}$$

according as $n > 0$ or $n < 0$. Instead of w_{ap} there is the analogous mean $\| \psi(\theta) \| = \| \psi(\theta) \|_{a,p} = \left(\int_0^{2\pi} |\psi(\theta)|^p |\theta|^{ap} d\theta \right)^{1/p}$, and the analogue of (1) is

$$(2) \quad A' \leq \|F\| / \|f\| \leq A''$$

when $F(\theta)^2 = \sum_n |\Delta_n(\theta)|^2$. The case $\alpha=0$ here is due to Littlewood and Paley [ibid. 42 (1936), 52-89]. This result is equivalent to (3) $\|T_\lambda f\| \leq A \|f\|$, where the transformation $T_\lambda f$ is defined by $T_\lambda f \sim \sum_n \varepsilon_n \Delta_n(\theta)$, and ε_n is a sequence with values ± 1 . The particular case $\varepsilon_n = 1$ for $n > 0$, $\varepsilon_n = -1$ for $n < 0$ gives $\|f\| \leq A \|f\|$, where f is the conjugate function of f . This result is due to Babenko [Dokl. Akad. Nauk SSSR (N.S.) 62 (1948), 157-160; MR 10, 249] and reduces to the theorem of M. Riesz [Math. Z. 27 (1927), 218-244] when $\alpha=0$.

The proof of (2) depends on the inequality

$$\int_{-\pi}^{\pi} \left[\int_0^1 |g(re^{i\theta})|^2 |1 - re^{i\theta}|^{2\alpha} (1-r) dr \right]^{p/2} d\theta \leq A \|g(e^{i\theta})\|^p$$

for functions $g(z)$ analytic in $|z| < 1$, a result which generalizes the case $\alpha=0$ of Paley and Littlewood. This proof does not simplify when $p=2$, and a special elementary proof is given for this. The inequality (3) is extended to transformation Λf defined by $\Lambda f(\theta) \sim \sum_n \lambda_n \varepsilon_n e^{in\theta}$ in which the sequence λ_n is bounded and of bounded variation in ranges $(\pm 2^k, \pm 2^{k+1})$, the case $\alpha=0$ being due to Marcinkiewicz [Studia Math. 8 (1939), 78-91]. H. R. Pitt.

Trigonometric Series and Integrals

Satô, Masako. Uniform convergence of Fourier series.

IV. Proc. Japan Acad. 31 (1955), 261-263.

Let $f(x)$ be continuous, of period 2π , and not constant. It is shown that

$$|s_n(x) - f(x)| \leq C \omega(1/n) \log \left(C_n \frac{\omega_1(1/n)}{\omega(1/n)} \right),$$

where $s_n(x)$ denotes the n th partial sum of the Fourier series, and $\omega(\delta)$, $\omega_1(\delta)$ are the modulus of continuity, or the integral modulus of continuity, of $f(x)$, respectively. The theorem follows in a few lines from former results of the author [same Proc. 30 (1954), 528-531, 698-701; MR 16, 692, 919], and so do other similar theorems of this paper on (C, α) -summability of the Fourier series.

W. W. Rogosinski (Newcastle-upon-Tyne).

Izumi, Shin-ichi. Some trigonometrical series. XII.

Proc. Japan Acad. 31 (1955), 207-209.

Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, with $\lim_{x \rightarrow \infty} a(x) = 0$. Let $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$ and $f(x) = \sum_{n=1}^{\infty} a_n \sin nx$. The author obtains asymptotic equalities for $f(x)$ and $f(x)$ as $x \rightarrow 0+$. Specimen results are as follows:

$$A_1 x \int_0^{1/x} t a(t) dt \leq f(x) \leq A_2 x \int_0^{1/x} t a(t) dt$$

if $f(x)$ is unbounded or $0 < B_1 \leq \int_0^{1/x} t a(t) dt \leq B_2 < \infty$, $0 < x < \infty$. If in addition $-a'(t)$ is convex and $\sum a_n = \infty$, then

$$A_3 \int_0^{1/x} t |a'(t)| dt \leq f(x) \leq A_4 \int_0^{1/x} t |a'(t)| dt.$$

P. Civin (Eugene, Ore.).

Izumi, Shin-ichi. Some trigonometrical series. XIII.

Proc. Japan Acad. 31 (1955), 257-260.

Let $s_n(x)$ be the n th partial sum of the Fourier series of the function $f(x)$ of class Lip α . The author shows that the series

$$\sum_{n=1}^{\infty} |s_n(x) - f(x)| / n^{1-2\alpha} (\log n)^\gamma$$

converges uniformly if $\alpha \leq \frac{1}{2}$ and $\gamma > 1$ or if $\alpha > \frac{1}{2}$ and $\gamma > 2$. P. Civin (Eugene, Ore.).

Izumi, Shin-ichi. Notes on Fourier analysis. XI. J.

Math., Tokyo 1 (1953), 80-86.

This paper consists of three independent parts. (I) Definitions of "quasi-orthogonality" have been given by Kac, Salem and Zygmund [Trans. Amer. Math. Soc. 63 (1948), 235-243; MR 9, 426] and by Gál and Koksma [Nederl. Akad. Wetensch., Proc. 53 (1950), 638-653 = Indag. Math. 12, 192-207; MR 12, 86]. The second class is known to include the first; the author shows that the first does not include the second. (II) According to Kaczmarz and Steinhaus,

$$L_n(t) = \int_0^1 \left| \sum_{k=1}^n \phi_k(t) \phi_k(u) \right| du = O\{n^{\frac{1}{2}} (\log n)^{\frac{1}{2} + \epsilon}\}$$

almost everywhere for an orthonormal $\{\phi_n\}$ on $(0, 1)$. The author proves that this still holds if orthonormality is replaced by

$$\int_0^1 \int_0^1 \left(\sum_{k=M+1}^{M+N} \phi_k(t) \phi_k(u) \right)^2 du = O(N).$$

(III) If $f_k(x)$ have period 1 and belong to L^2 , if $\phi(t) \downarrow 0$ as $t \downarrow 0$, and if $\int_0^1 |f_k(x) - f_k(x+u)|^2 dx \leq M \{\phi(u)\}^2$ and $\sum \{\phi(1/n_k)\}^2 < \infty$, then

$$n_k^{-1} \sum_{i=1}^{n_k} f_k(x + i/n_k) - \int_0^1 f_k(x) dx \rightarrow 0;$$

in particular, the case $f_k(x) = f(a_k x + b_k)$, $a_k = O(1)$ gives a result on convergence of Riemann sums.

R. P. Boas, Jr. (Evanston, Ill.).

Matsuyama, Noboru, and Takahashi, Shigeru. On the convergence of some gap series. Proc. Japan Acad. 31 (1955), 111-115.

Let $f(x)$ have period 1, $\int_0^1 f dx = 0$, $\int_0^1 f^2 dx = 1$. The author generalizes results of Kac, Salem and Zygmund [Trans. Amer. Math. Soc. 63 (1948), 235-243; MR 9, 426] and Izumi [Tôhoku Math. J. (2) 3 (1951), 89-103; MR 14, 868] dealing with convergence of series $\sum c_n f(n x)$, with a lacunary sequence $\{n_k\}$, to cases where $\{n_k\}$ is $\lambda(n)$ -lacunary in the sense of Alexits [Acta Univ. Szeged. Sect. Sci. Math. 13 (1949), 14-17; MR 10, 701].

R. P. Boas, Jr. (Evanston, Ill.).

Men'šov, D. E. On almost convergent trigonometric series. Mat. Sb. N.S. 37(79) (1955), 265-292. (Russian)

It is well known that, if a sequence $f_0(x), f_1(x), \dots, f_n(x), \dots$ of functions measurable and finite almost everywhere on an interval (a, b) converges in measure to a function $f(x)$, finite almost everywhere in (a, b) , then a) from $\{f_n\}$ we can select a subsequence $f_{n_1}, f_{n_2}, \dots (n_1 < n_2 < \dots)$ converging to f almost everywhere in (a, b) ; b) if some sequence $f_{n_1}, f_{n_2}, \dots (n_1 < n_2 < \dots)$ converges to a finite limit $\varphi(x)$ in a set $E \subset (a, b)$, then $\varphi(x) = f(x)$ almost everywhere in E . There exist sequences $\{f_n\}$ satisfying a) and b) but not converging in measure.

The author calls a trigonometric series

$$(*) \quad T = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

almost convergent on $(-\pi, \pi)$ to $f(x)$ if the partial sums $s_n(x)$ of T are almost convergent to $f(x)$. It is shown that 1) given any three measurable functions $f_i(x)$ ($i=1, 2, 3$) finite almost everywhere in $(-\pi, \pi)$ we can represent any trigonometric series $(*)$ as a sum of three trigonometric series, $T = T_1 + T_2 + T_3$, such that T_i is almost convergent to f_i ($i=1, 2, 3$). Furthermore, 2) if T is finite in measure (i.e., if the s_n are uniformly bounded outside a set of measure as small as we please), then the series T_i in 1) can be so chosen that

$$\limsup_{n \rightarrow \infty} |a_{in}| \leq \limsup_{n \rightarrow \infty} |a_n|, \quad \limsup_n |b_n| \leq \limsup_n |b_n| \quad (i=1, 2, 3),$$

whose a_{in}, b_{in} are the coefficients of T_i ; in particular, 3) if T has coefficients tending to 0, the T_i can be chosen so as to have coefficients tending to 0. *A. Zygmund.*

Sinhal, S. D. Borel summability of the conjugate series of a derived Fourier series. *Duke Math. J.* 22 (1955), 445-450.

A series $\sum a_n$ is said to be summable to s by the Borel method if the limit

$$\lim_{r \rightarrow +\infty} e^{-r} \sum_{n=0}^{\infty} r^n s_n / n!$$

exists and equals s , where $s_n = a_0 + a_1 + \dots + a_n$.

Let $f(t)$ be an integrable function with period 2π and let $\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$. The author then proves that if $\varphi(t) = o(t)$ as $t \rightarrow 0$ and $\varphi(t)/t$ is of bounded variation on the right of $t=0$, then the series conjugate to the derived Fourier series of $f(x)$ is summable to

$$\lim_{r \rightarrow +\infty} (2\pi)^{-1} \int_{1/r}^{\pi} \varphi(t) (\sin(t/2))^{-2} dt$$

by the Borel method at a point x where the above integral exists. *S. Izumi (Tokyo).*

Chow, H. H. On the summability for positive indices of the Fourier series of a function with an infinite limit. *Acta Math. Sinica* 5 (1955), 81-89. (Chinese. English summary)

Let $\sigma_n^r(\theta)$ be the r th Cesàro mean of

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

and

$$\phi(t) = \frac{1}{2}[f(\theta+t) + f(\theta-t)].$$

If $0 < r < 1 - \delta$ ($0 < \delta < 1$), there exists a non-monotone $\phi(t)$ with $\lim_{t \rightarrow \infty} \phi(t) = \infty$ and $\liminf_{n \rightarrow \infty} \sigma_n^r(0) = -\infty$.

K. L. Chung (Syracuse, N. Y.).

Alexits, G. Sur la caractérisation de certaines classes de fonctions au sens de la théorie constructive des fonctions. *Acta Math. Acad. Sci. Hungar.* 6 (1955), 41-46. (Russian summary)

Let $\sigma_n(x)$ denote the (C, α) partial sums of the Fourier series of $f(x)$, and let $f, \tilde{\sigma}_n^{\alpha}$ denote the conjugate function and its (C, α) partial sums. The author proves the following theorems. (1) A necessary condition for f to belong to $\text{Lip}(1, p)$, $p \geq 1$, is that for every positive α there exists

$F_{\alpha} \in L^p$ such that

$$n|f(x) - \sigma_n^{\alpha}(x)| \leq F_{\alpha}(x) \quad \text{and} \quad n|f(x) - \tilde{\sigma}_n^{\alpha}(x)| \leq F_{\alpha}(x);$$

and either of these inequalities, with one α , is sufficient for f to belong to $\text{Lip}(1, p)$. (2) If $f'(x_0)$ exists, $f(x_0) - \sigma_n^{\alpha}(x_0) = O(1/n)$. (3) If $f \in \text{Lip}(1, p)$ in (a, b) , $p > 1$, then for every positive ε there is an $F_{\varepsilon} \in L^p$ such that $n|f(x) - \tilde{\sigma}_n^1(x)| \leq F_{\varepsilon}(x)$ in $(a+\varepsilon, b-\varepsilon)$.

R. P. Boas, Jr. (Evanston, Ill.).

Longuet-Higgins, M. S. Bounds for the integral of a non-negative function in terms of its Fourier coefficients. *Proc. Cambridge Philos. Soc.* 51 (1955), 590-603.

The author's problem is to find bounds for $\int_E f(\theta) d\theta$, when $f(\theta)$ is an unknown nonnegative function whose first $2N+1$ (real) Fourier coefficients are given, and E is a given set. He is interested in the case where f is zero outside neighborhoods of not more than N points. To solve the problem he considers

$$I(\alpha_1, \dots, \alpha_N) = \pi^{-1} \int_0^{2\pi} f(\theta) g(\theta) d\theta,$$

where $g(\theta)$ is a trigonometric polynomial of degree N which is positive except at $\theta = \alpha_1, \dots, \alpha_N$, where it vanishes, and chooses the α 's to make I small. The results are illustrated by examples. In his discussion the author uses the well-known determinant criterion for the trigonometric moment problem to have a solution which is a step-function with N jumps, and he gives a new and quite direct proof of this. The problem arose in trying to infer from observations the angular distribution of the energy of ocean waves. *R. P. Boas, Jr.*

Shapiro, Victor L. The Laplacian of Fourier transforms. *Duke Math. J.* 22 (1955), 435-444.

The paper discusses the summability by spherical means [cf. Bochner, *Trans. Amer. Math. Soc.* 40 (1936), 175-207] of the Laplacian of the Fourier transform of a function. If I_R is the integral of a function over the interior of a sphere of radius R , and $S_R = 2\pi R^{-2} \int_0^R (R^2 - r^2)^{n-1} r dr$ tends to a limit as $R \rightarrow \infty$, then the integral of the function is said to be spherically (C, α) summable.

If

$$\lim_{T \rightarrow \infty} \lim_{R \rightarrow \infty} \frac{n}{T^n} \int_0^R \exp(-r^n T^{-n}) r^{n-1} I_r dr$$

exists, the integral is said to be spherically (A, n) summable to its limit; and a function f is said to be of exponential type of order $(n-1)$ if $f(x) = o(\exp -a|x|^n)$ as $|x| \rightarrow \infty$ for every $a > 0$.

The main theorems of the paper are that: if g is Lebesgue summable over Euclidean k -space E_k , and its Fourier transform f is twice differentiable over every bounded domain in E_k , then the Fourier integral of $(2\pi)^{-k} f(x)$ is spherically summable (C, δ) and (A, n) to $-|u|^2 g(u)$ almost everywhere, for every $\delta > \max(\frac{1}{2}(k-1), 1)$ and for every positive integer n ; (2) if g is in $L^2(E_k)$, the integral is spherically summable (C, η) to $-|u|^2 g(u)$ for every $\eta > \max(k/2, 3/2)$; and if f is of exponential type of order $(n-1)$ it is also spherically (A, n) summable to $-|u|^2 g(u)$ almost everywhere.

As an application, the following uniqueness theorem is proved: if a trigonometrical integral $\int \exp i(x \cdot u) c(u) du$ is spherically summable to $f(x) \in \text{Lip } \alpha$ everywhere save on a set of capacity zero, and $c(u)/(|u|^2 + 1) \in L$, then

$$(2\pi)^{-k} f(x) \exp(-iu \cdot x) dx$$

is (C, δ) summable to $c(u)$ almost everywhere.

J. L. B. Cooper (Cardiff).

See also: Hyltén-Cavallius, p. 247; Zaring, p. 253; Yano, p. 257; Hirschman, p. 257; Bochner, p. 273.

Integral Transforms, Operational Calculus

Hsu, L. C. Note on generalized Jordan's condition for the Fourier and Mellin transforms. *Acta Math. Sinica* 3 (1953), 142-147. (Chinese. English summary)

"...inversion theorems for the Fourier and Mellin transforms are established under more general conditions in which the term 'of bounded variation' is used in the sense of Saks." (From the author's abstract.)

K. L. Chung (Syracuse, N.Y.).

Hsu, L. C., and Wu, Chih-Chuan. Concerning a generalized Stieltjes-Post inversion formula and an asymptotic integral. *Acta Math. Sinica* 5 (1955), 161-172. (Chinese. English summary)

Some extensions of previous results by the first author [Amer. J. Math. 73 (1951), 199-210; Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5 (1949), 273-279; MR 12, 497; 13, 215].

K. L. Chung (Syracuse, N.Y.).

Freud, G. Restglied eines Tauberschen Satzes. III. *Acta Math. Acad. Sci. Hungar.* 5 (1954), 275-289.

[For parts I and II see same *Acta* 2 (1951), 299-308; 3 (1952), 299-307; MR 14, 361, 958.] The main theorem deduces asymptotic appraisals of Riesz-Stieltjes transforms of nonnegative integer orders from a given asymptotic appraisal of a Laplace-Stieltjes transform and a Tauberian hypothesis. Let $\tau^*(t)$ have bounded variation over each finite interval $0 \leq t \leq t_0$, and let

$$F^*(s) = \int_0^\infty e^{-st} d\tau^*(t)$$

exist when $s > 0$. Let, as $s \rightarrow 0+$,

$$F^*(s) = A + O\{R(s)\}$$

where $R(s)$ is increasing over $s \geq 0$, $R(0+) = 0$, and $R(ks) < e^{\alpha k} R(s)$ when $s > 0$ and $k = 2, 3, 4, \dots$. Let $\beta_2(t)$ be a monotone increasing function and let L be a positive constant such that, as $s \rightarrow 0+$,

$$\int_0^\infty e^{-st} d\beta_2(t) = \frac{1}{s} [1 + O\{R(s)\}]$$

and the function $\gamma_2(t)$ defined by

$$\gamma_2(t) = L\beta_2(t) + \int_0^t u d\tau^*(u)$$

is a monotone increasing function of t . Then, for each $m = 0, 1, 2, \dots$,

$$\int_0^\infty (x-t)^m d\tau^*(t) = A \frac{x^m}{m!} \left[1 + O\left\{ \left(\log \frac{1}{R(1/x)} \right)^{m-1} \right\} \right]$$

as $x \rightarrow \infty$. Related theorems and applications to Dirichlet series and power series are given. R. P. Agnew.

Sargent, W. L. C. On the transform $\gamma_a(s) = \int_0^\infty x(t) k_a(t) dt$. *J. London Math. Soc.* 30 (1955), 401-416.

Six theorems give characterizations of kernels $k_a(t)$ such that, under a specified method for evaluating integrals, the transformation set forth in the title carries each function $x(t)$ defined everywhere or almost everywhere over $t \geq 0$ and belonging to a stated class A into a function $\gamma_a(s)$ defined everywhere over $s \geq 0$ and belonging to a

stated class B . In the first three theorems, a particular Denjoy (general or special) method is used; A is the class of functions $x(t)$ integrable over $t \geq 0$; and B is the class of functions $y(s)$ defined when $s \geq 0$ and (Theorem 1) measurable; (Theorem 2) measurable and essentially bounded; (Theorem 3) Lebesgue integrable. For Theorems 1 and 2, the conditions characterizing $k_a(t)$ are relatively simple; Theorem 3 involves one less palatable condition and the proofs of necessity and sufficiency of the conditions are both more complicated. In Theorem 4, f_0^∞ is defined to be $\lim_{h \rightarrow \infty} \int_0^h f_0^\infty$ where f_0^∞ is, when $h < \infty$, a Lebesgue integral; A is the class of functions $x(t)$ integrable over $t \geq 0$ in this sense; and B is the class of functions $y(s)$ bounded over $t \geq 0$. Theorem 6 is similar to Theorem 4.

Theorem 5 is the only theorem involving a quantitative relation between $\gamma_a(s)$ and $x(t)$. It gives necessary and sufficient conditions that a kernel $k_a(t)$ be such that

$$\gamma_a(s) = \int_0^\infty x(t) k_a(t) dt = \lim_{h \rightarrow \infty} \int_0^h x(t) k_a(t) dt$$

exists for each $s \geq 0$ and

$$(*) \quad \lim_{s \rightarrow \infty} \gamma_a(s) = \int_0^\infty x(t) dt = \lim_{h \rightarrow \infty} \int_0^h x(t) dt$$

whenever integrals over finite intervals are Lebesgue integrals and $x(t)$ is such that the right side of (*) exists and is finite. This result is of substantial interest because it gives conditions analogous to the two familiar conditions ($\sum_{k=0}^\infty |a_{n,k} - a_{n,k+1}| < M$ for each n , and $\lim_{n \rightarrow \infty} a_{n,k} = 1$ for each k) which characterize matrices $a_{n,k}$ such that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^\infty a_{n,k} u_k = \sum_{k=0}^\infty u_k$$

whenever $\sum u_k$ converges. The conditions on $k_a(t)$ are the following three. (i) There exists a nonnegative function $a(s)$ such that, for each $s \geq 0$, $k_a(t)$ is measurable and essentially bounded over $0 \leq t < a(s)$, and $k_a(t)$ has bounded variation over the infinite interval $t \geq a(s)$. (ii) There exist nonnegative constants a , s_0 , and M such that, for each $s > s_0$, $k_a(t)$ is essentially dominated by M over $0 \leq t < a$ and $k_a(t)$ has total variation less than M over the infinite interval $t \geq a$. (iii) For each $\lambda > 0$,

$$\lim_{s \rightarrow \infty} \int_0^s [1 - k_a(t)] dt = 0.$$

The author points out that her three conditions on $k_a(t)$ are not obtainable from conditions given by G. G. Lorentz [*Acta Math.* 79 (1947), 255-272; MR 9, 278] who solved a similar problem involving Stieltjes transformations of the form $\gamma_X(s) = \int_0^\infty f(t) dX(t)$. R. P. Agnew (Ithaca, N.Y.).

Timan, M. F. Integral transforms of a function of two variables. *Soobšč. Akad. Nauk Gruzin. SSR* 15 (1954), 135-142. (Russian)

The integral $\iint_Q f(x, y) dx dy$ of $f(x, y)$ over the quadrant $x, y \geq 0$ is evaluable to I by the Cesàro (or Riesz) method $C(\alpha, \beta)$ if

$$\lim_{u, v \rightarrow \infty} \int_0^u \int_0^v f(u, v) \left(1 - \frac{u}{x}\right) \left(1 - \frac{v}{y}\right)^\beta dx dy = I,$$

and is evaluable to I by the Abel method A if

$$\lim_{u, v \rightarrow \infty} \int_0^u \int_0^v f(u, v) e^{-xu} e^{-vy} du dv = I.$$

Some relations are obtained between these methods for assigning values to integrals. R. P. Agnew.

Chakravarty, N. K. On some theorems and inequalities in operational calculus with two variables. *Bull. Calcutta Math. Soc.* 46 (1954), 221-235.

The author evaluates a number of double Laplace transforms, and obtains formally some "rules" and inequalities for such transforms. He attempts to justify interchanging the order of integrations, etc., in the derivation of his formulas, but the conditions under which he states that interchanging the order of integrations "will be taken to hold" are in fact neither necessary nor sufficient for this. While his formulas are probably correct under certain conditions, these conditions cannot be extracted from the paper. *A. Erdélyi*.

Chakravarty, N. K. On certain theorems in operational calculus with n variables. *Bull. Calcutta Math. Soc.* 46 (1954), 259-264.

Formal derivation of four "rules" of n -dimensional operational calculus. *A. Erdélyi* (Pasadena, Calif.).

Bose, S. K. Laplace transform and self-reciprocal functions. *Ganita* 5 (1954), 25-32.

The author proves five results of which one of the two simplest is as follows: "If $\phi(p) \neq f(t)$, $p^{1/2}/(1/p) \neq g(t)$, and $t^{-n-1/2}g(t/2)$ is self-reciprocal in the Hankel transform of order $n-1$, then $t^{n-3/2}\phi(t)$ is self-reciprocal in the Hankel transform of order n , provided $\text{Re } n > -\frac{1}{2}$, the integrals are absolutely convergent, differentiation [of a certain relation involving two infinite integrals] is permissible and the integrals [in another such relation] are absolutely convergent." *A. Erdélyi* (Pasadena, Calif.).

Kumar, Ram. A self-reciprocal function. *Ganita* 5 (1954), 53-59.

The author shows that a certain function involving two generalized hypergeometric series is self-reciprocal in the Hankel transformation. *A. Erdélyi* (Pasadena, Calif.).

Bhatnagar, K. P. On self-reciprocal functions and a new transform. *Bull. Calcutta Math. Soc.* 46 (1954), 179-199.

[See also MR 14, 977; 15, 216, 790; 16, 468.] The author investigates the kernel

$$w_{\mu}(xy) = (xy)^{\frac{1}{2}} \int_0^{\infty} J_{\nu}(t) J_{\mu}(xy/t) dt/t,$$

which is the resultant of two Hankel kernels, and gives a number of self-reciprocal and skew-reciprocal functions for this kernel. He also states a number of general lemmas which he regards as extensions of Lerch's theorem, but many of his proofs are unconvincing or defective. *A. Erdélyi* (Pasadena, Calif.).

Bhatnagar, K. P. On self-reciprocal functions involving two complex variables. *Ganita* 5 (1954), 33-44.

In this paper functions are obtained which are self-reciprocal in the w_{μ} -transform, and also relations connecting self-reciprocal functions for two w_{μ} kernels. [The reader should beware of misprints and typographical imperfections.] The principal tool is the Mellin transform. *A. Erdélyi* (Pasadena, Calif.).

Bhatnagar, K. P. Some self-reciprocal functions. *Bull. Calcutta Math. Soc.* 46 (1954), 245-250.

A list of self-reciprocal functions for w_{μ} and for the similar resultant $w_{\mu\nu}$ of several Hankel kernels. No proofs are given. *A. Erdélyi* (Pasadena, Calif.).

Bhatnagar, K. P. A general theorem. *Bull. Calcutta Math. Soc.* 46 (1954), 251-252.

A formula connecting self-reciprocal functions for two different kernels $w_{\mu\nu}$. *A. Erdélyi* (Pasadena, Calif.).

Bose, B. N. Certain theorems on self-reciprocal relationship in operational calculus. I. *Bull. Calcutta Math. Soc.* 46 (1954), 141-152.

The object of the present note is to obtain generalizations of some of the theorems obtained in earlier papers [B. N. Bose, same *Bull.* 44, 93-110 (1953); MR 15, 120; S. C. Mitra and B. N. Bose, *Acta Math.* 88 (1952) 227-240; MR 14, 555]. [Reviewer's remark: The generalizations are simple corollaries of the older results.] *A. Erdélyi*.

Bose, B. N. Certain theorems on self-reciprocal relationship in operational calculus. II. *Bull. Calcutta Math. Soc.* 46 (1954), 201-215.

Extension of the results of two earlier papers [same *Bull.* 44 (1952), 93-110; 46 (1954), 109-127; MR 15, 120; 16, 585]. *A. Erdélyi* (Pasadena, Calif.).

See also: San Juan Llosá, p. 139; Slater, p. 150; Stanković, p. 164; Hewitt and Rubin, p. 172; Bochner, p. 273; Eberlein, p. 281.

Special Functions

Riekstyn's, Ē. Ya. On a polynomial applicable to the solution of telegraph equations. *Prikl. Mat. Meh.* 18 (1954), 738-744. (Russian)

The problem of finding the inverse Laplace transform of $e^{-\lambda p} p^{-1} (p + \alpha - \beta)^n (p + \alpha + \beta)^n$ leads the author to study the polynomial

$$La_n(t, \lambda) = - \sum_{k=0}^n \frac{t^k}{k!} \sum_{r=k+1}^{n+1} (-1)^r \binom{n+1}{r} \lambda^r,$$

which for $\lambda=1$ reduces to a Laguerre polynomial. A number of recurrence relations and connections with Laguerre polynomials are established. These polynomials are then used to solve the problem of a single telegraph line, with given e.m.f. at one end, earthed through a resistance and inductance at the other, with constant R, C, L, G subject to $RC=LG$. This involves estimating the magnitude of the polynomials for large t . There is apparently an application also to vibrating strings but no details are given. *F. V. Atkinson* (Canberra).

Singh, Vikramaditya. Appell polynomials of large order. *J. London Math. Soc.* 30 (1955), 475-479.

Starting from the integral representation $P_n(x) = \int_0^1 (x+t)^n/n! d\beta(t)$ for Appell polynomials, the author obtains the asymptotic relation

$$P_n(x/n) n! x^{n-n} = \sum_{m=0}^{\infty} (-1)^m n^{-m} \sum_{l=0}^m C_{m-l}^l (1/l!) x^{l+m} (d/dx)^{l+m} A(x).$$

Here $A(t)$ is the generating function for $\{P_n(x)\}$: $A(t)e^{xt} = \sum_{n=0}^{\infty} t^n P_n(x)$, and the C 's are given by $(\frac{1}{2} + \frac{1}{2}\omega + \dots)^m = \sum_{l=0}^m C_l^m \omega^l$. Particular cases (Hermite, Euler, Bernoulli polynomials) are examined, and a theorem on expansions in Appell polynomials given. *I. M. Sheffer*.

Jepsen, Donald W., Haugh, Eugene F., and Hirschfelder, Joseph O. The integral of the associated Legendre function. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 645-647.

The authors evaluate in closed form $\int_{-1}^1 P_n^m(x) dx$ for non-negative integers m and with $n \geq m$. A. Erdélyi.

Sarkar, G. K. On integral representations of the generalized k -function of Bateman and its connection with Legendre and parabolic cylinder functions. *Bull. Calcutta Math. Soc.* 46 (1954), 239-244.

The author transcribes some known formulas involving confluent hypergeometric functions in the notation of "generalized k -functions", and manipulates these formulas. A. Erdélyi (Pasadena, Calif.).

Saran, Shanti. Integrals associated with hypergeometric functions of three variables. *Proc. Nat. Inst. Sci. India. Part A.* 21 (1955), 83-90.

The author obtains integral representations for ten hypergeometric functions of three variables. One type of integral representation is of the form

$$\int t^{x-1}(1-t)^{y-1}/(x, y, z, t) dt,$$

where f is a hypergeometric function of one or two variables, or a product of such hypergeometric functions, and the contour of integration is a Pochhammer-Jordan double loop; and other integral representations are of the Mellin-Barnes type. A. Erdélyi (Pasadena, Calif.).

Ragab, F. M. Some formulae for the product of two Whittaker functions. *Nederl. Akad. Wetensch. Proc. Ser. A.* 58=Indag. Math. 17 (1955), 430-434.

The author evaluates some infinite integrals the integrands of which contain a product of two confluent hypergeometric functions multiplied by other functions. A. Erdélyi (Pasadena, Calif.).

Henrici, Peter. On certain series expansions involving Whittaker functions and Jacobi polynomials. *Pacific J. Math.* 5 (1955), 725-744.

The following expansion theorem is proved. Let $z = x + iy$ and $z^* = x - iy$, where x and y are complex variables. Given $r > 0$, consider the region B of (x, y) -space for which $|z| < r$ and $|z^*| < r$. Suppose $F(z, z^*)$ is a solution of

$$(*) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{4\mu+1}{x} \frac{\partial u}{\partial x} + \frac{4\nu+1}{y} \frac{\partial u}{\partial y} + k[4\lambda - k(x^2+y^2)]u = 0$$

which is regular in B and is an even function of x and y . Then if $F(z, 0) = \sum_{n=0}^{\infty} a_n z^{2n}$, $F(z, z^*) = \sum_{n=0}^{\infty} a_n F_n^{(\mu, \nu)}(z, z^*)$ in B , where the functions $F_n^{(\mu, \nu)}(z, z^*)$ are certain products of Whittaker functions and Jacobi polynomials (J.-W. functions) which satisfy (*). A host of special cases of this expansion are of interest.

By considering special solutions of (*) in (x, y) -coordinates, one is led to the series of J.-W. functions generated by

$$[e(1+\tau)/2]^{-\mu-1} M_{\mu, \nu}[e(1+\tau)/2] \cdot [e(1-\tau)/2]^{-\nu-1} M_{\mu, \nu}[e(1-\tau)/2],$$

the M 's denoting the usual Whittaker functions. As the author remarks, this is a mother expansion having a large number of familiar children and grandchildren. Among these are three examples of expansions for the

product of two Bessel functions and the identity of Ramanujan,

$${}_1F_1(a; c; z) \cdot {}_1F_1(a; c; -z) = {}_2F_3(a, c-a; c, c/2, (c+1)/2; z^2/4).$$

If one uses the orthogonality of the Jacobi polynomials, one also obtains from this generating-function relation a generalization of an integral representation of Erdélyi [H. Buchholz, Die konfluente hypergeometrische Funktion, Springer, Berlin, 1953, p. 128; MR 14, 978].

By considering solutions of (*) in Jacobian elliptic coordinates, an addition theorem analogous to theorems of Graf and Gegenbauer [see G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed., Cambridge, 1944, p. 358; MR 6, 64] is found as another special case of the expansion theorem. Lastly, a generating function (a product of two ${}_2F_1$'s) of Brafman [Proc. Amer. Math. Soc. 2 (1951), 942-949; MR 13, 649] and an identity of Bailey expressing a ${}_4F_3$ as the product of two ${}_2F_1$'s [Generalized hypergeometric series, Cambridge, 1935] are derived.

There are several misprints, the more vital of which are: p. 725, lines 18 and 19, for E read E_B ; in formula (3) for z^2 read y^2 ; the formula immediately preceding Lemma 1 is (18); p. 729, line 21, for E read E_B ; p. 731, line 5, for E read E_D ; in formula (37) in the first ${}_1F_1$ for $-z^2/4$ read $z^2/4$; in formula (42), 2nd line, the upper indices of P_m are $(2\nu, 2\mu)$ not $(2\mu, \nu)$ and in the first index of M for $\mu+\nu+\frac{1}{2}$ read $\alpha+\nu+\frac{1}{2}$; in formula (65), 1st line, for $+\tilde{\omega}$ read $-\tilde{\omega}$. N. D. Kazarinoff (Lafayette, Ind.).

Romberg, Werner. Die Forderung des Reihenabbrechens zur Eigenwertbestimmung. *Norske Vid. Selsk. Forh., Trondheim* 28 (1955), 62-66.

The author shows how to construct, using the Laplace transform, a solution of the Weber equation, $\psi'' + (2\nu+1-x^2)\psi = 0$, which vanishes as $x \rightarrow \infty$ for each ν . He thus illustrates that the boundary condition $\lim_{x \rightarrow \infty} \psi = 0$ is insufficient to determine discrete eigenvalues. He then shows that the eigenfunctions are determined by the additional condition that $\lim_{x \rightarrow -\infty} \psi = 0$. The author fails to remark that these results follow immediately from the known behavior of Weber functions [H. Buchholz, Die konfluente hypergeometrische Funktion, Springer, Berlin, 1953, p. 38 et seq.; MR 14, 978]. N. D. Kazarinoff (Lafayette, Ind.).

Vidyasagar, G. C. On certain integrals. *Ganita* 5 (1954), 61-68.

"The object of the present note is to evaluate certain integrals with the help of Operational Calculus. The integrand in several cases involves Hypergeometric Functions". (Author's introduction.) Unfortunately, it is difficult to see which new integrals the author succeeded in evaluating. A. Erdélyi (Pasadena, Calif.).

Ordinary Differential Equations

Leontovič, E., and Maier, A. On a scheme determining the topological structure of the separation of trajectories. *Dokl. Akad. Nauk SSSR (N.S.)* 103 (1955), 557-560. (Russian)

Let (1) $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ where P, Q are in $C^{(1)}$ in a closed bounded plane region \bar{G} . The boundary of \bar{G} is assumed to be a simple closed curve consisting of a finite

number of trajectories of (1) and a finite number of arcs without contact. A "singular" trajectory or semi-trajectory is one which is orbitally unstable or which passes through the common end of a boundary arc without contact and an arc of a trajectory. [Cf. Leontovich and Mayer, C.R. (Dokl.) Acad. Sci. URSS (N.S.) 14 (1937), 251-254; and Markus, Trans. Amer. Math. Soc. 76 (1954), 127-148; MR 15, 704.] The critical points of (1) are to be isolated and only a finite number of singular curves are assumed to exist. Theorem: In order that the topological structure of two differential systems of the form (1) be the same it is necessary and sufficient that their schemes of separation by singular trajectories be isomorphic. The definitions of "topological equivalence" and "isomorphism" are not very explicit in the paper.

L. Markus (Princeton, N.J.).

Gomory, R. E., and Haas, F. A study of trajectories which tend to a limit cycle in three-space. Ann. of Math. (2) 62 (1955), 152-161.

Let $S: \dot{x}_i = f_i(x_1, x_2, x_3)$ ($i=1, 2, 3$), where f_i are real analytic in E^3 . Let C be a limit cycle and let S be a solution of S tending towards C as $t \rightarrow \infty$. There exists an analytic homeomorphism of $E^3 - C$ onto the exterior of a torus T in E^3 . Then S is carried onto a differential system S' which can be extended to the surface T [cf. Gomory, Ann. of Math. (2) 61 (1955), 140-153; MR 16, 700]. The authors use the Denjoy theory of differential equations on the torus T to obtain information on the limiting directions of S as it approaches C . Excepting the case in which S' on T has a finite number of periodic solutions all of which are semi-stable, the authors show that for each surface of section centered on C , S has either a finite number of limiting directions or else every direction is a limiting direction for S . L. Markus (Princeton, N.J.).

Mikeladze, S. E. Remarks on the theory of discontinuous solutions of ordinary differential equations. Soobšč. Akad. Nauk Gruz. SSR 15 (1954), 647-654. (Russian)

Let $Ly=0$ be a linear ordinary differential equation with continuous coefficient. The author has given [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 55 (1947); 789-792; MR 9, 36] an expression in terms of a fundamental set of solutions for a solution with jump discontinuities in $y^{(k)}$ ($k=0, \dots, n-1$). Here he first extends this expression to cover the case (P) in which the coefficients depend also continuously on an arbitrary number of parameters. Next he discusses a boundary-value problem for a self-adjoint second-order equation, concluding that in case (P) the proper values of the parameters cannot be determined, for a solution with assigned discontinuities, without supplementary conditions. The final section treats special equations soluble in closed form. F. A. Ficken.

Volosov, V. M. Quasihomogeneous differential equations of the second order having a small parameter. Mat. Sb. N.S. 36(78) (1955), 501-554. (Russian)

Previous results of the author [Dokl. Akad. Nauk SSSR (N.S.) 73 (1950), 873-876; Mat. Sb. N.S. 30(72) (1952), 245-270; 31(73) (1952), 645-674; MR 12, 101; 14, 276, 1086] are here generalized to equations $\mu(xy'' + \beta y' + \gamma) + Q(y, x) = 0$ ($\mu > 0$). Assume that the functions α, β, γ, Q are defined and continuous for $x_0 \leq x \leq \bar{x}$, $-\infty < y, y', y'' < +\infty$, Q has piecewise-continuous second derivatives, satisfies $|y - f(x)| \leq Q \leq M|y - f(x)|$, where f is defined and has a bounded second derivative in $[x_0, \bar{x}]$; $Q \leq 0$ when $y \leq f$.

$\alpha = \alpha(y'', y', y, x)$ satisfies $0 < C_1 \leq \alpha \leq C_2$, has limits $\alpha_i(y', y, x)$ ($i=1, 2$), corresponding to $y \leq f$, $y'' \rightarrow +\infty$ and $y \geq f$, $y'' \rightarrow -\infty$, $|\alpha - \alpha_i| \leq K|y'|^{-k}$, $k > 1$, 2; α_i have limits $\alpha_i^0(y, x)$ when $|y'| \rightarrow \infty$, $|\alpha_i - \alpha_i^0| \leq K|y'|^{-l}$ ($l > 1$); the α_i are continuous and the α_i^0 have piecewise-continuous second derivatives. $\beta = \beta(y'', y', y, x)$ is bounded, has uniform limits $\beta^k(y'', y, x)$ ($k=1, 2$), corresponding to $y' \rightarrow \pm\infty$; each one of the β^k has uniform limits $\beta_i^k(y, x)$ when $y'' \rightarrow \pm\infty$, $y \leq f$; the β^k are continuous and the β_i^k have piecewise-continuous first derivatives. $\gamma = \gamma(y'', y', y, x)$ satisfies $|\gamma| \leq \varphi(y)$, φ being any even function. Then, if $y(x, \mu)$ is the solution satisfying $y(x_0) = y_0$, $y'(x_0) = y_0'$, and if $y_0 = f(x_0)$, $y(x, \mu) \rightarrow f(x)$ uniformly when $\mu \rightarrow 0$; if $y_0 \neq f(x_0)$, $y(x, \mu)$ is bounded and oscillates around $y = f(x)$ with a frequency of the order of μ^{-1} , the maxima and minima approaching certain curves $y = F_k(x)$, $k=1, 2$, which satisfy certain differential equations and whose initial values $F_k(x_0)$ depend on the value of $y_0 - f(x_0)$. If $\varphi(y', y, x)$ is any bounded continuous function satisfying the same assumptions as β , the limit as $\mu \rightarrow 0$ of $\int_a^b \varphi(y'', y', y, x) dx$ ($x_0 \leq a < b \leq \bar{x}$), exists and may be explicitly calculated. Several examples show that most of the assumptions made are essential; if they are not satisfied $y(x, \mu)$ may diverge. The limit cases $k=\frac{1}{2}$ and $l=1$ in the assumptions on α are also considered; in these cases, with a certain refinement of the assumptions, the main results still hold true. J. L. Massera.

Nikolenko, L. D. On oscillation of solutions of the differential equation $y'' + p(x)y = 0$. Ukrain. Mat. Ž. 7 (1955), 124-127. (Russian)

It is shown that a necessary condition for $y'' + p(x)y = 0$ to have solutions which are oscillatory as $x \rightarrow \infty$ is $\int_0^\infty x \log x \cdot \max\{p(x) - \frac{1}{4}x^{-2}, 0\} dx = \infty$. He regards this as "near" to the sufficient condition of Gagliardo [Boll. Un. Mat. Ital. (3) 8 (1953), 177-185; MR 15, 126]. [Reviewer's remark: The author's test is the second of a chain of tests which may be derived by a change of variables starting from the test $\int_0^\infty x \max\{p(x), 0\} dx = \infty$, in analogy to the chain of tests given by J. C. P. Miller, Proc. Cambridge Philos. Soc. 36 (1940), 283-287; MR 2, 50.] F. V. Atkinson (Canberra).

Nemyckii, V. V. Estimate of the regions of asymptotic stability of nonlinear systems. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 803-804. (Russian)

Let $\dot{x} = Ax + f(t, x)$, x, f being n -vectors, $A = (a_{ik})$ a matrix; assume that the real parts of the characteristic roots of A are $\leq -b^2$, $\|f\| \leq C_R \|x\|$ in the sphere $\|x\| \leq R$, and that the conditions for the existence of solutions are satisfied in this sphere; let $m \leq n$ be the number of non-identically-vanishing components of f , $\|A\| = (\sum a_{ik}^2)^{1/2}$. Then, if $\varepsilon = n^{-2}(n-1)^{1/2}b^2\|A\|^{-1}$, $C_R < b^2n^{-2}m^{-1}\varepsilon^{n-1}$, the positive half-trajectories starting from points satisfying $\|x\| \leq \varepsilon^{n-1}R$ lie in the sphere $\|x\| \leq R$. If the matrix A is triangular or diagonal, better bounds are found. A sketch of the proof is given. It is claimed that these results are simpler than those given by I. G. Malkin [Prikl. Mat. Meh. 16 (1952), 495-499; MR 14, 170] and P. V. Atrasenok [Vestnik Leningrad. Univ. 9 (1954), no. 8, 79-106]. J. L. Massera (Montevideo).

Sestini, Giorgio. Criterio di stabilità in un problema non lineare di meccanica dei sistemi a più gradi di libertà. Riv. Mat. Univ. Parma 5 (1954), 227-232.

A result by the author concerning the stability of scalar second-order differential equations [same Riv. 2

(1951), 303-314; MR 14, 170, 1277] is extended to vectorial equations of the form

$$\ddot{P}(t) = -G(P) - R(P, \dot{P}) + F(t).$$

Here $P(t)$ is a vector with components $x_i(t)$ ($i=1, \dots, n$), and G, R, F are vectors whose components are denoted by g_i, r_{ij}, f_i ($i=1, \dots, n$) respectively. The following assumptions are made: (a) $g_i x_i \geq 0$, $g_i \geq \omega^2 x_i$ (ω a positive constant); (b) the r_{ij} are bounded for bounded x_i and for all x_i , and $r_{ij} x_i \geq 0$, $\lim_{|x_i| \rightarrow \infty} r_{ij} = \infty$; (c) the $f_i = f_i(t)$ are of bounded variation in some interval $t_0 \leq t < \infty$. It is then proved that all integrals of the differential equation that are defined in (t_0, ∞) are there stable, i.e. the corresponding functions $|x_i(t)|$, $|\dot{x}_i(t)|$ are bounded. The proof is short and elementary. *W. Wasow* (Los Angeles, Calif.).

McLachlan, N. W. Two theorems on ordinary non-linear differential equations. *Math. Gaz.* 39 (1955), 200-202.

A strong stability theorem is proved for the equation

$$\ddot{y} + g(y)\dot{y} + f(y) = 0,$$

showing that under certain restrictions on f and g , y and \dot{y} tend to zero as t (the independent variable) tends to infinity. This is applied to a mechanical system and to the problem of designing a hydro-electric reservoir and surge tank so as to ensure that oscillations in the flow are damped. In a second theorem the first theorem is applied to show that under suitable conditions on f the solution of the system

$$\ddot{y} + 2k\dot{y} + f(y) = 0$$

together with its derivative tend to zero in a non-oscillatory manner. *E. Pinney* (Berkeley, Calif.).

Marcus, M. Some results on the asymptotic behavior of linear systems. *Canad. J. Math.* 7 (1955), 531-538.

The linear system is $\dot{x} = C(t)x$, where x is a complex n -vector and $C(t)$ is an n -square complex matrix function continuous on $[0, \infty]$. Two cases are studied: (1) $C(t) = A + B(t)$, where A is a constant matrix and the conditions on B , somewhat weaker than absolute integrability, imply that $B(t)$ is almost diagonal as $t \rightarrow \infty$; (2) $C(t) = A(t) + C(t)$, where the $A_{ij}(t)$ are almost periodic and the $B_{ij}(t)$ are absolutely integrable (i.e., the $C_{ij}(t)$ are almost almost-periodic as $t \rightarrow \infty$). For case (1) two types of sufficient conditions are given for the boundedness of all solutions on $[0, \infty]$ and for the convergence of all solutions to the origin as $t \rightarrow \infty$. For case (2) sufficient conditions are given for the boundedness of all solutions on $[0, \infty]$.

J. P. LaSalle (Notre Dame, Ind.).

Bellman, Richard. Functional equations in the theory of dynamic programming. II. Nonlinear differential equations. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 482-485.

[For part I see same Proc. 39 (1953), 1077-1082; MR 15, 887.] This paper provides a dynamic-programming approximation method for the solution of a nonlinear system and a quasi-linear system of differential equations. Existence and uniqueness theorems are stated.

T. L. Saaty (Washington, D.C.).

Atkinson, F. V. On asymptotically linear second-order oscillations. *J. Rational Mech. Anal.* 4 (1955), 769-793.

The author derives asymptotic formulae of the form $x(t) = A \cos(\int_0^t \omega dt + B) + o(1)$ as $t \rightarrow \infty$ for bounded so-

lutions of $\ddot{x} + xF(x, t) = 0$, which is to be asymptotically linear and oscillatory in the sense that $F(x, t) \rightarrow 1$ as $t \rightarrow \infty$ for any fixed x . In the asymptotic formula A and B are constants and ω a function of t and the asymptotic amplitude A . Under certain conditions of smoothness of the function $F(x, t)$ there are proved three asymptotic formulae. The first, which is the simplest, has the form $x(t) = A \cos\{\int_0^t (2V)^{-1/2} x F dt + B\} + o(1)$, where $V(x, t) = \int_0^x \xi F(\xi, t) d\xi$. This result constitutes an explicit formula only in the linear case. *M. Zlámal* (Brno).

Atkinson, F. V. On second-order non-linear oscillations. *Pacific J. Math.* 5 (1955), 643-647.

The author considers the differential equation (1) $y'' + f(x)y^{2n-1} = 0$, where n is an integer ≥ 2 and $f(x)$ is positive and continuous for $x \geq 0$. He proves that a necessary and sufficient condition for all solutions of (1) to be oscillatory is that $\int_0^\infty x f(x) dx = \infty$. (This condition is known to be necessary but not sufficient when $n=1$). He shows also that if $f'(x)$ is continuous and $f'(x) \leq 0$ for $x \geq 0$, a sufficient condition for nonoscillation of all solutions $\neq 0$ is $\int_0^\infty x^{2n-1} f(x) dx < \infty$. *W. Leighton*.

Urabe, Minoru. Infinitesimal deformation of the periodic solution of the second kind and its application to the equation of a pendulum. *J. Sci. Hiroshima Univ. Ser. A.* 18 (1954), 183-219.

The system $\dot{x} = P_1(x, \theta, \epsilon)$, $\dot{\theta} = Q_1(x, \theta, \epsilon)$ is studied in a way similar to that in a previous paper [same J. 18 (1954), 37-53; (1955), 401; MR 17, 37] but it is now assumed that P_1 and Q_1 are periodic in θ ; as before, it is almost impossible to summarize the results. The special case of the equation of a pendulum $\ddot{\theta} + \epsilon f(\theta)\dot{\theta} + g(\theta) = 0$ is considered in great detail. *J. L. Massera* (Montevideo).

Šimanov, S. N. On a method of obtaining conditions for the existence of periodic solutions of nonlinear systems. *Prikl. Mat. Meh.* 19 (1955), 225-228. (Russian) Consider

$$(1) \dot{z}_s = \sum_{\lambda=1}^n a_{s\lambda}(z) z_\lambda + Z_s(t, z_1, \dots, z_n, \mu) \quad (s=1, \dots, n),$$

where the Z are power series in z_1, \dots, z_n, μ convergent in the neighborhood of $z_1 = \dots = z_n = \mu = 0$ and containing neither constant nor linear terms; the coefficients of these series as well as the $a_{s\lambda}$ are continuous periodic functions of t of period 2π . The problem is to find small harmonic oscillations corresponding to small nonvanishing values of μ , in the critical case where the linear system $\dot{z}_s = \sum a_{s\lambda} z_\lambda$ has vanishing characteristic exponents. By means of a nonsingular linear transformation with constant or periodic coefficients the system may be reduced to

$$(2) \dot{x}_{1i} = X_{1i}(t, x, y, \mu), \dot{x}_{vi} = x_{vi-1} + X_{vi}(t, x, y, \mu),$$

$$\dot{y}_j = \sum_{i=1}^l c_{ji} y_i + Y_j(t, x, y, \mu)$$

$$(i=1, \dots, k; v_i=2, \dots, m_i; j=1, \dots, l),$$

c_{ji} constants, X, Y satisfying similar assumptions as the Z . Consider the system (3) obtained by the addition of indeterminate constants W_i to the first k equations of (2). Then (3) has small harmonic vibrations satisfying the initial conditions $x_{vi}(0) = \beta_{vi}$ ($v_i=1, \dots, m_i$), $y_j(0) = \gamma_j$, where $\beta_i = \beta_{m_i+1}$ are arbitrary and $\beta_{vi} = \beta_{vi}(\mu, \beta_i)$ ($v_i=1, \dots, m_i-1$), $\gamma_j = \gamma_j(\mu, \beta_i)$, $W_i = W_i(\mu, \beta_i)$ certain analytic functions. A necessary and sufficient condition for the existence of small harmonic vibrations of (1) is that

the system $W_i(\mu, \beta_i) = 0$ have solutions $\beta_i = \beta_i(\mu)$ vanishing with μ . The method may be applied practically by using formal series for x, y, W . J. L. Massera (Montevideo).

Obi, Chike. Uniformly almost periodic solutions of nonlinear differential equations of the second order. I. General exposition. Proc. Cambridge Philos. Soc. 51 (1955), 604-613.

The author considers the differential equation

$$(*) \quad \ddot{x} + \chi(x) = k_1 f_1(x, \dot{x}) + k_2 \chi(x) + eg(x, \dot{x}, t + \beta)$$

in which he assumes, among other things, that $\chi(x)$, f_1 and g are analytic, that $f_1 \geq 0$ and that g is uniformly almost periodic (u.a.p.) in t with $(\lambda_1, \lambda_2, \dots, \lambda_N)$ as a set of basic exponents. Let $x = x(\alpha, \beta, k_1, k_2, \epsilon, t)$ be the solution of (*) such that $x - \alpha = \dot{x} = 0$ when $t = 0$ and let $x(\alpha, \beta, 0, 0, 0, t) = x_0$. By the author's hypotheses x_0 is periodic and may be written as $x_0 = \sum_{r=0}^{\infty} a_r \cos r\phi t$. Define

$$y_0 = \sum_{r=0}^{\infty} (da_r/d\alpha) \cos r\phi t.$$

The principal result in the present paper is a pair of "characteristic equations" which for small ϵ , k_2 and k_1 are necessary and sufficient conditions for the solution x to be u.a.p. The conditions are

$$m(\dot{x}_0/2) = 0 \text{ and } m\left(y_0/2 - \frac{1}{\epsilon} \frac{d\phi}{d\alpha} \int_0^t x_0/2 dt\right) = 0$$

where $m(\cdot) = \lim_{T \rightarrow \infty} T^{-1} \int_0^T (\cdot) dt$ and $f_2 = k_1 f_1(x, \dot{x}) + k_2 \chi(x) + eg(x, \dot{x}, t + \beta) + \chi(x_0) - \chi(x) + \chi'(x_0)(x - x_0)$. C. E. Langenhof (Ames, Iowa).

Graffi, Dario. Su alcune equazioni differenziali non lineari. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) 1 (1954), no. 1, 57-64.

The author establishes some basic facts concerning the solutions of the equation

$$\ddot{x} + \dot{F}(x) + [\omega_0^2 + g(x) \sin(\Omega t + \gamma)]x = 0.$$

Assumptions: (1) ω_0^2 , Ω and γ are constants, the first two positive. (2) $F'(x)$ and $g(x)$ are continuous, and satisfy Lipschitz conditions in each finite interval. (3) There exists a positive number Δ such that if $|x| > \Delta$, $F(x)$ has the same sign as x , and $|F(x)|$ increases monotonically without limit as $|x|$ increases. (4) There exists a positive number N such that

$$(\omega_0^2 + |g(x)|) \cdot |x| < N|F(x)|, |x| > \Delta.$$

It is shown that any solution which is defined for $t = 0$ is continuable over the infinite interval $t > 0$. Under the further assumption: (5) $F'(0) < 0$, $g(0) < \omega_0^2$, $xg(x) > 0$ for $|x|$ sufficiently large, and $|g(x)|$ increases with $|x|$, it is shown that every solution has infinitely many maxima and minima. If, in addition, $|g(x)| < \omega_0^2$, every solution has infinitely many zeros.

The case in which $g(x)$ is a constant m and $|m| < \omega_0^2$ gives what is called the mixed equation of Liénard and Mathieu. In this case between any two successive zeros of a solution there lies at least one zero of any solution of the Mathieu equation obtained by setting $F(x) = 0$, $g(x) = m$. This fact leads to further information about the zeros of the solutions of the given equation. L. A. MacColl.

Caprioli, Luigi. Su un modello meccanico per le oscillazioni di rilassamento. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) 1 (1954), no. 2, 152-168.

The author studies the motions of a mechanical system consisting of two containers mounted on a pivoted bar so

that each container is capable of taking a lower position and an upper position. When a container is in its upper position it receives liquid at a given rate, and when it is in its lower position it discharges liquid at a given rate. The study shows that several different types of motion are possible, depending on the values of various parameters entering into the specification of the system. Some of the motions exemplify the relaxation oscillations considered in the theory of nonlinear vibrations. L. A. MacColl.

Vogel, Theodore. Breaking oscillations in servo systems. J. Mental Sci. 100 (1954), 103-113.

This is a non-technical expository paper on the phase-plane analysis of nonlinear systems. The oscillations referred to in the title are oscillations which sometimes occur in situations where the governing differential equations change their analytical forms when the moving representative phase point meets certain critical curves. L. A. MacColl (New York, N.Y.).

Pinney, Edmund. Nonlinear differential equations. Bull. Amer. Math. Soc. 61 (1955), 373-388.

The author discusses a method for the solution of nonlinear differential equations in which the nonlinearities do not occur in the most highly differentiated terms. First the method is applied to van der Pol's equation and carried through in detail. The van der Pol equation is equivalent to the integral equation

$$y(t) = y(0) \cos t + y'(0) \sin t + \epsilon \int_0^t \sin(t-\tau) [1 - y^2(\tau)] y'(\tau) d\tau.$$

When we write $y(t) = a_+(t)e^{it} + a_-(t)e^{-it}$, where

$$a_{\pm}(t) = \frac{1}{2} y(0) \pm \frac{1}{2i} y'(0) \pm \frac{\epsilon}{2i} \int_0^t e^{\pm i\tau} [1 - y^2(\tau)] y'(\tau) d\tau,$$

we obtain

$$(*) \quad a_{\pm}'(t) = \frac{1}{2} \epsilon [(1 - a_+ a_-)(a_{\pm} - a_{\mp} e^{\mp 2it}) - a_{\pm}^3 e^{\pm 2it} + a_{\mp}^3 e^{\pm 4it}].$$

By the well known device, which consists in neglecting the oscillatory terms on the right-hand side of (*), the quantities $a_{\pm}(t)$ are approximately equal to the solution of the system $a_{\pm}' = \frac{1}{2} \epsilon (1 - a_{\pm} a_{\mp}) a_{\pm}$, that is to $r(t)e^{i\phi}$, where $r(t) = [1 + C e^{-t}]^{-1/2}$ and C and ϕ are real constants. The magnitude of the error is examined and the final result is

$$y(t) = 2r(t) \cos \{[1 + O(\epsilon^2)]t + \phi\} + O(\epsilon),$$

$$y'(t) = -2r(t) \sin \{[1 + O(\epsilon^2)]t + \phi\} + O(\epsilon).$$

The same method is then applied to systems of nonlinear differential equation and to a partial differential equation occurring in a nonlinear transmission-line problem. For brevity results only are given. M. Zlámál (Brno).

Butenin, N. V. On the theory of forced synchronization. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 187-195. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

Following the method of Andronov and Witt the author studies the stability of harmonic oscillations for the equation $\ddot{\phi} + \phi = \mu[-\phi + e(\phi) + f_0 \sin t]$, where $\mu > 0$ is a small parameter and $e(\phi) = \pm e_0$ according as $\phi \leq 0$.

H. A. Antosiewicz (Washington, D.C.).

Andronov, A. A., and Bautin, N. N. On the influence of Coulomb friction in the dashpot on processes of indirect regulation. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1955, no. 7, 34-48. (Russian)

Mathematically speaking this study reduces to the investigation of a system of three linear equations dis-

continuous along certain planes. The trajectories of two such special systems are studied and described piecewisely at considerable length; and much detailed information regarding these systems is given. [Reference: Andronov, Bautin, and Gorelik, *Avtomat. i Telemekh.* 7 (1946), 15-41.]
S. Lefschetz (Princeton, N.J.).

Ulanov, G. M. On the theory of dynamic precision of a nonlinear system of indirect regulation. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 465-467.

This note contains a brief sketch of some results on control processes, together with references to earlier papers where more complete results are contained.

R. Bellman (Santa Monica, Calif.).

Vinograd, R. È. Instability of the smallest characteristic exponent of a control system. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 541-544. (Russian)

The author discusses the two systems $x'=0$, $y'=f(t)y$, and $x'=ay$, $y'=ax+f(t)y$, $a>0$, as an example of the instability of the smallest characteristic exponent as a approaches zero.

R. Bellman.

Letov, A. M. Stability of unsteady motions of control systems. Prikl. Mat. Meh. 19 (1955), 257-264. (Russian)

Let

$$\dot{\eta}_k = \sum_{j=1}^n b_{kj}\eta_j + n_k \xi, \quad \dot{\xi} = f(\sigma), \quad \sigma = \sum_{j=1}^n p_{kj}\eta_j - \xi,$$

where the b , n , p are functions of t , f continuous except perhaps for $\sigma=0$, $f(0)=0$, $\sigma[f(\sigma)-h\sigma]>0$ for $\sigma \neq 0$, $h>0$. Let $\Delta=(a_{pq})$ be any symmetric positive definite matrix, S any positive number. A sufficient condition for stability is that the form $W=\sum B_{ss}\zeta_s\zeta_s+2\sum Q_{ss}\zeta_s\zeta_s+(\varrho+h)\zeta_s^2$ be positive definite, where the B , Q and ϱ are certain very complicated expressions explicitly given as functions of the b , n , p and the a , S . J. L. Massera (Montevideo).

Mitrinovich, Dragoslav S. Sur l'équation différentielle d'Emden généralisée. C. R. Acad. Sci. Paris 241 (1955), 724-726.

It is shown that the equation

$$f(x)y''+g(x)y'+ay^n=0 \quad (a \text{ const.})$$

has a solution $y=h(x)u(x)$, where u is a solution of the first-order equation

$$f h^{1-n} u'^2 + A u^2 + 2a u^{n+1}/(n+1) = C$$

provided f and g have the form

$f(x)=(Bh^2-A)h^{n+1}/h'^2$, $g(x)=Ah/h'-(Bh^2-A)h^{n+1}h''/h'^3$, for some function h , A , B , and C are constants. This result is specialized further and plans for future work are mentioned.

E. Pinney (Berkeley, Calif.).

Godeaux, Lucien. Sur une équation différentielle linéaire. Mathesis 64 (1955), 81-87.

It is shown that the substitution $y=z \exp(-\frac{1}{2}ax^2)$ reduces the differential equation

$$y^{(n)} + \binom{n}{1}axy^{(n-1)} + \dots$$

$$+ \binom{n}{p}a^2x^2y^{(n-p)} + \dots + a^nx^ny=0 \quad (n \geq 2)$$

to a linear differential equation with constant coefficients. Some simple particular cases are discussed in detail.

L. A. MacColl (New York, N.Y.).

Harazov, D. F. On boundary problems in the theory of ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) 100 (1955), 217-220. (Russian)

The author announces results on the distribution of the eigen-values of the problem $F(y)=G(y; \lambda)-f(x)$, where F and G are linear self-adjoint ordinary differential operators of orders $2n$ and $2m < 2n$, considered over a finite interval with boundary conditions of general type; here F is to be positive or negative definite, and G is also a polynomial in λ of degree t . Assuming a knowledge of the Green's function for F and the boundary conditions in question, the problem can be reduced to the operator form $w - \sum_{k=0}^t \lambda^k A_k w = -f$; deductions can then be made from postulates concerning the A_k . A typical result is: If t is odd, and A_0 has no positive eigen-values ≤ 1 , then the problem has at least one real eigen-value. After stating four more such results, further investigations are summarized for the case when the system reduces to the operator form $w - \lambda A_1 w - \lambda^2 A_2 w = -f$. For example, if A_2 has only negative eigen-values, then the problem has at least one real eigen-value. Assuming further that A_2 has only positive eigen-values, the author states an expansion theorem giving the solution of the inhomogeneous problem, which also seems to hold if $A_2=0$. The work of which this is a continuation was published in same Dokl. (N.S.) 91 (1953), 1023-1026, 1285-1287; MR 15, 881.

F. V. Atkinson (Canberra).

Crum, M. M. Associated Sturm-Liouville systems. Quart. J. Math. Oxford Ser. (2) 6 (1955), 121-127.

If ϕ_s, λ_s ($s=0, 1, \dots$) are the eigen-functions and eigen-values of a Sturm-Liouville problem, then $\phi_{s+1}=\phi_s'-\phi_0\phi_s/\phi_0$, λ_s ($s=1, 2, \dots$) are those of another such problem, the first associated system. Repeating the process, the n th associated system is found, and this is given here explicitly in determinantal form. Special consideration is given to the end-points, which are in general singular for the associated systems, and a special argument is therefore given to prove completeness. The trigonometric, Hermite, Legendre and Hankel cases are used as examples. An incidental result is the proof of a Sturmian oscillatory property [see J. Liouville, J. Math. Pures Appl. 1 (1836), 253-265, 269-277]. The reverse process is also discussed; it can be used to make the spectrum contain an assigned finite set of eigen-values. The general inverse Sturm-Liouville problem is however not treated.

F. V. Atkinson (Canberra).

Slobodyanskii, M. G. Approximate solution of a self-adjoint boundary-value problem for an ordinary differential equation and determination of the regions of distribution of eigenvalues. Prikl. Mat. Meh. 18 (1954), 585-596. (Russian)

The first part of the paper is concerned with the determination of bounds for the eigenvalues of the self-adjoint problem

$$(*) \quad Au - \lambda u = \sum_{k=0}^n (-1)^k \frac{d^k}{dx^k} \left[p_k(x) \frac{d^k u}{dx^k} \right] = f(x) \quad (a < x < b),$$

$$u^{(s-1)}(a) = u^{(s-2)}(a) = \dots = u(a) = 0;$$

$$u^{(s-1)}(b) = u^{(s-2)}(b) = \dots = u(b) = 0,$$

making use of previous results of the author's [Dokl. Akad. Nauk SSSR (N.S.) 91 (1953), 733-736; Prikl. Mat. Meh. 17 (1953), 623-626; MR 17, 195]. The second part of the paper deals with the estimation of bounds for the solution of the boundary-value problem arising from (*) upon putting $\lambda=0$. Numerical bounds are given in the

particular example:

$$-\frac{d}{dx}\left(\frac{du}{dx}\right) + (\beta + x)u = 1 \quad (0 < x < 1), \quad u(0) = u(1) = 0,$$

where β is a constant. *J. B. Diaz* (College Park, Md.).

See also: Romberg, p. 262; Slobodyanskii, p. 286; Moisil, p. 320.

Partial Differential Equations

Haimovici, M. On the completion of Pfaffian systems. *Com. Acad. R. P. Române* 1 (1951), 757-758. (Romanian)

If S is a Pfaffian system of r linear equations $\omega^i = 0$ with characters s_1, \dots, s_p in involution with respect to ϕ of the independent variables, the author seeks the condition that $S + \omega$ where ω is a single Pfaffian, be in involution with respect to those variables. He announces: (i) the existence of an integer m such that $\omega^i(d_j, d)$ forms with $\omega^j(d), \omega^j(d_j, d)$ ($j=1, \dots, m-1$) a set of Pfaffians linear in d on which $\omega^i(d_m, d)$ are dependent; (ii) the inequality $m < s_p + 2$. Proofs are to be given elsewhere.

J. M. Thomas (Durham, N.C.).

Mihăilescu, Tiberiu. Sur l'intégration d'une équation Riccati généralisée. *Acad. Repub. Pop. Romîne. Fil. Cluj. Stud. Cerc. Şti.* 5 (1954), no. 1-2, 37-44. (Romanian. Russian and French summaries)

The equation considered is $dz = \omega_1 z^2 + \omega_2 z + \omega_3$, where $\omega_1 = \omega_2 = \omega_3 = 0$ is a passive Pfaffian system. If the rank of the Pfaffian system is three, it is shown that z is found without integration and involves two arbitrary functions of one variable. The solution is also carried out in particular cases where two of the ω 's are proportional.

J. M. Thomas (Durham, N.C.).

Nicolau, Edmond. Une propriété des systèmes différentiels auto-adjoints. *Acad. Repub. Pop. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz.* 6 (1954), 903-911. (Romanian. Russian and French summaries)

For a linear system of partial differential equations with constant coefficients the following results are established: (i) The conservation relation $C(u)$ of each self-adjoint system is additive in the sense $C(u+v) = C(u) + C(v)$. (ii) There are systems which have no conservation relation. (iii) There are non-self-adjoint systems with additive conservation relations. Further there are non-linear systems with non-additive conservation relations.

J. M. Thomas (Durham, N.C.).

Sun, J. Tseying. On the canonical form of a system of linear partial differential equations. *J. Chinese Math. Soc. (N.S.)* 1 (1951), 36-67. (Chinese summary)

The system consists of a linear part $\partial x^i / \partial x^k + a_{ik} x^j = 0$ and a quadratic $g_{ij} x^i x^j = \text{constant}$ with $\det g_{ij} \neq 0$. The range of i equals that of j but not necessarily that of k and the summation convention is used. The a 's and g 's are known functions of the independent variables x . By transformation of the unknown x 's the system is reduced to a form in which the range of j is reduced in each equation to a subset of the original range. For an ordinary system ($k=1$) that range for the i th equation holds at most two values $i-1, i+1$ and exactly one of these for the extreme values of i . This case includes the Frenet formulas.

J. M. Thomas (Durham, N.C.).

Moisil, Gr. C. Les préliminaires algébriques des théorèmes d'existence. *Com. Acad. R. P. Române* 1 (1951), 341-343. (Romanian. Russian and French summaries)

This paper concerns solutions of the linear system $\sum_{i=1}^r P_{ijk}(x) \partial u^j / \partial x^k = 0$ on a given variety defined by r equations among the independent variables. It is asserted that induced systems which permit the construction of existence theorems can be formed and that in the case of constant P 's finding the induced systems reduces to calculations with certain polynomials. *J. M. Thomas*.

Gel'fand, I. M., and Šilov, G. E. On a new method in uniqueness theorems for solution of Cauchy's problem for systems linear partial differential equations. *Dokl. Akad. Nauk SSSR (N.S.)* 102 (1955), 1065-1068. (Russian)

Préliminaires: soit $S(\alpha, \beta; A, B)$ l'espace des fonctions ϕ indéfiniment différentiables sur R (α, β, A, B positifs) telles que pour tout $\varepsilon, \delta > 0$, il existe $N_{\varepsilon, \delta}(\phi) < \infty$ avec

$$|x^k \phi^{(n)}(x)| \leq N_{\varepsilon, \delta}(\phi) (A + \delta)^k k^{2n} (B + \varepsilon)^n n^{\alpha \beta},$$

pour tout x, k et n ; topologie naturelle. Lemme: soit $f(s) = \sum_{n=0}^{\infty} a_n s^n$, une fonction entière d'ordre $\leq 1/\beta$, de type $\leq \beta/B^{1/\beta} e^{\alpha s}$; alors l'opérateur différentiel infini $f(D)$, $D = d/dx$, est un opérateur linéaire continu de $S(\alpha, \beta; A, B)$ dans $S(\alpha, \beta; A, B e^{\alpha})$. Généralisation des espaces et du lemme à N variables.

Application: on considère le problème de Cauchy pour le système d'évolution

$$(*) \quad \frac{\partial}{\partial t} - P\left(\frac{1}{i} \frac{\partial}{\partial x}, t\right),$$

P étant une matrice carrée (m, m) , dont les coefficients sont des opérateurs différentiels linéaires d'ordre q sur R^N , $x \in R^N$, à coefficients indépendants de x , continus en t . On associe à (*) le système $d/dt - P(s, t)$, de matrice fondamentale $Q(s, t_0, t)$, fonction entière de s , d'ordre $q_0 \leq q$ [cf. Gel'fand et Šilov, *Uspehi Mat. Nauk (N.S.)* 8 (1953), no. 6 (58), 3-54; MR 15, 867]. Le lemme (dans R^N) et les raisonnements usuels [Schwartz, *Ann. Inst. Fourier, Grenoble* 2 (1951), 19-49; MR 13, 242] donnent l'existence et l'unicité du problème de Cauchy dans le dual de $S_0^q = \bigcup_{A, B} S(\alpha, \beta; A, B)$ ($\alpha \geq 1 - 1/q_0$). *J. L. Lions*.

Kostyučenko, A. G. On a uniqueness theorem for solution of Cauchy's problem and of a mixed problem for certain types of systems of linear partial differential equations. *Dokl. Akad. Nauk SSSR (N.S.)* 103 (1955), 13-16. (Russian)

On désigne par (I) le résumé précédent. On considère le système $\partial/\partial t - P(\partial/\partial x, x)$, $x \in R^N$, P désignant une matrice carrée (m, n) dont les coefficients sont des opérateurs différentiels linéaires qui cette fois sont à coefficients variables en x (mais indépendants de t). On introduit l'espace $PS(\alpha, \beta; A, B)$ (analogue à (I)) des fonctions vectorielles ϕ indéfiniment différentiables sur R^N , telles que

$$|x^k P(\partial/\partial x, x)^n \phi(x)| \leq C_{\varepsilon, \delta}(\phi) (A + \delta)^k k^{2n} (B + \varepsilon)^n n^{\alpha \beta},$$

P étant l'adjoint de P . On suppose que cet espace contient „assez de fonctions”. Lemme comme dans (I): $\sum_{n=0}^{\infty} t^n P^n / n!$ est un opérateur continu (t fixé) sur $PS(\alpha, \beta; A, B)$, A et B convenables. Application au problème de Cauchy comme dans (I).

On considère ensuite $P(\partial/\partial x, x)$ défini sur un ouvert \mathfrak{M} de R^N (problème mixte); $PS_{\mathfrak{M}}^q$ est l'espace des ϕ

définies sur \mathbb{R} avec $|\bar{F}^n \varphi| \leq C(\varepsilon)(A + \varepsilon)^n n^n$. Applications (brièvement indiquées) à certains problèmes mixtes.

Exemples intéressants. Notamment, pour

$$\partial/\partial t - i(\Delta - q(x)), \quad x \in \mathbb{R}^N,$$

q bornée ainsi que toutes ses dérivées, il y a unicité du problème de Cauchy si l'on cherche des solutions majorées (pour tout t) par $C_1 \exp C(x_1^2 + x_2^2 + x_3^2)$. J. L. Lions.

Žautykov, O. A. On Cauchy's problem for a denumerable system of partial differential equations of the first order. Izv. Akad. Nauk Kazah. SSR 1952, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 81-87. (Russian. Kazak summary)

The author gives an existence and uniqueness theorem, and a well-posedness theorem, for the following Cauchy problem:

$$\frac{\partial u_s}{\partial x} - \lambda_s(x, y) \frac{\partial u_s}{\partial y} = f_s(x, y, u_1, u_2, \dots),$$

$$u_s(x, y)|_{x=0} = \varphi_s(y) \quad (s=1, 2, \dots).$$

J. B. Diaz (College Park, Md.).

Kamynin, L. I. On uniqueness of solution of Cauchy's problem in a class of rapidly increasing functions for an infinite system of ordinary equations. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 545-548. (Russian)

In earlier papers [same Dokl. (N.S.) 82 (1952), 13-16; 93 (1953), 397-400; 95 (1954), 13-16; Izv. Akad. Nauk SSSR. Ser. Mat. 17 (1953), 163-180; MR 14, 172; 16, 251, 524; 14, 1090] the author has considered the question of the uniqueness of Cauchy's problem for a system of the form

$$(a) \quad \frac{\partial u}{\partial t}(x, t) = f(x, t; u(x+h_{-1}, t), \dots, u(x+h_n, t))$$

$$(u(x, 0) = \varphi(x), \quad -\infty < x < +\infty, \quad |t| \leq T).$$

In these papers, various classes of admissible functions were discussed, for which the Cauchy problem (a) is uniquely solvable, but it was supposed throughout that f is bounded in absolute value with respect to x . In the present paper it is shown that if the right-hand term of (a) does not increase with x faster than $|x|^{1-\delta}$, then (a) has a unique solution in the class of (admissible) functions satisfying

$$|u(x, t)| \leq A \exp \left\{ (1-\varepsilon) \frac{|x|}{H} \ln(1+|x|) \right\}$$

$$(H = \max_{|t| \leq T} |h_n|, \quad |t| \leq T),$$

where ε and δ are connected by the inequality $\delta > 1 - \varepsilon \geq 0$. J. B. Diaz (College Park, Md.).

Harazov, D. F. On the investigation of boundary problems for elliptic differential equations. Dokl. Akad. Nauk SSSR (N.S.) 100 (1955), 421-424. (Russian)

The boundary problem is $\Lambda(u) + L(u; \lambda) = f(x, y)$, where $\Lambda(u)$ is a linear self-adjoint elliptic differential operator of order $2n$ in the variables x, y , and $L(u; \lambda)$ is another such operator of lower order which is a polynomial in λ , considered in a finite region on the boundary of which there hold conditions $U_i(u) = 0$ ($i=1, \dots, n$), where the U_i are linear differential operators of order $< 2n$. Assuming a knowledge of the Green's function for Λ and the boundary conditions, this problem is put in the operator form $w - \sum_{k=0}^n \lambda^k A_k w = -f$. Results are then stated which

are practically the same as those announced for the ordinary case [same Dokl. (N.S.) 100 (1955), 217-220; MR 17, 266]. F. V. Atkinson (Canberra).

Daugavet, I. K. Application of the general theory of approximate methods to the investigation of convergence of Galerkin's method for certain boundary-value problems of mathematical physics. Dokl. Akad. Nauk SSSR (N.S.) 98 (1954), 897-899. (Russian)

The boundary-value problem under consideration is

$$(*) \quad \Delta x - \lambda a(T)x = y(T), \quad T \in B; \quad x(T) = 0, \quad T \in \Gamma;$$

where Δ is the Laplacian, B is a bounded open set of m -dimensional Euclidean space (of points $T = (t_1, t_2, \dots, t_m)$) with smooth boundary Γ , and the constant λ is not an eigenvalue. The author considers the convergence of Galerkin's method when applied to the solution of this problem, basing himself on the general theory of approximate methods for linear problems due to L. V. Kantorovič [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 89-185; MR 10, 380]. Galerkin's method consists in seeking a solution of (*) in the form

$$x_n = \sum_{k=0}^n \sum_{i_1+\dots+i_m=k} c_k^{(i_1, \dots, i_m)} \varphi_k^{(i_1, \dots, i_m)}(T),$$

where

$$\varphi_k^{(i_1, \dots, i_m)}(T) = \Omega(T) t_1^{i_1} t_2^{i_2} \dots t_m^{i_m},$$

and the function $\Omega(T)$ is such that: (1) $\Omega(T)$ vanishes on Γ but never is zero on B ; (2) $\text{grad } \Omega(T) \neq 0$ for T on Γ ; (3) $\Omega(T)$ is sufficiently differentiable on $B + \Gamma$. The constant coefficients $c_k^{(i_1, \dots, i_m)}$ are determined by the system

$$\int_B (\Delta x_n - \lambda a x_n) \varphi_i^{(i_1, \dots, i_m)} dB$$

$$= \sum_{k=0}^n \sum_{i_1+\dots+i_m=k} c_k^{(i_1, \dots, i_m)} \int_B (\Delta \varphi_k^{(i_1, \dots, i_m)} - \lambda a \varphi_k^{(i_1, \dots, i_m)}) \varphi_i^{(i_1, \dots, i_m)} dB = \int_B y \varphi_i^{(i_1, \dots, i_m)} dB$$

$$(i=0, 1, \dots, n; \quad i_1+i_2+\dots+i_m=i),$$

where $\{\varphi_i^{(i_1, \dots, i_m)}\}$ is a fixed sequence of functions.

J. B. Diaz (College Park, Md.).

Ėidus, D. M. On a boundary problem for the equation $\Delta u + \lambda^2 u = 0$. Dokl. Akad. Nauk SSSR (N.S.) 100 (1955), 631-633. (Russian)

Let Ω be a simply connected region in 3-space bounded by a Liapounoff surface S . It is shown that if $A/f(x) = \int_S (\sin |x-y|) |x-y|^{-1} f(y) dS_y = 0$ on S , then f is the normal derivative of a solution u of $\Delta u + \lambda^2 u = 0$ in Ω , which vanishes at the boundary. If $\{\varphi_k\}_{k=1}^\infty$ is a complete orthonormalized set of eigenfunctions of A , considered as an operator on $L^2(S)$, and λ^2 is not an eigenvalue of $-A$, then the corresponding eigenvalues $\{\mu_k\}_{k=1}^\infty$ of A are positive and $u = \sum (\varphi_k, u) \mu_k^{-1} A \varphi_k$ is the unique solution of $\Delta u + \lambda^2 u = 0$ in Ω such that $u = \varphi$ on the boundary.

L. Gårding (Lund).

Rubinowicz, A. Über eine Verallgemeinerung des Reziprozitätstheorems für Lösungen der Schwingungsgleichung mit Multipolquellen. Acta Phys. Polon. 14 (1955), 183-190. (Russian summary)

The author derives a generalization of the reciprocity theorem associated with the equation $\Delta u + k^2 u = 0$.

A. E. Heins (Pittsburgh, Pa.).

Burčuladze, T. V. On the asymptotic distribution of the eigenfunctions for vibration of an elastic body. Soobšč. Akad. Nauk Gruz. SSR 15 (1954), 193-200. (Russian).

Let $e(t, x, x')$ be the spectral function associated with the second boundary-value problem for a bounded elastic body B . The asymptotic formula and (1) $\text{tr } e(t, x, x) = c^{2n/3}(1+o(1))$, where c is a certain constant, is shown when B is bounded by a Liapounoff surface. Under more restrictive conditions, it is due to A. Pleijel [Ark. Mat. Astr. Fys. 27A (1940), no. 13; MR 2, 291]. [Reviewer's remark. (1) can, also for B not bounded and any self-adjoint boundary condition, be replaced by $e(t, x, x') = e_0(t, x-x') + O(t)$, where e_0 is the spectral function for the entire space; cf. Gårding, Kungl. Fysiogr. Sällsk. i Lund Förh. 24, no. 21; (1955), MR 17, 158.] L. Gårding.

Sandgren, Lennart. A vibration problem. Medd. Lunds Univ. Mat. Sem. 13 (1955), 1-84.

Il s'agit d'un problème de vibration suggéré par une question d'hydrodynamique: l'étude des oscillations dans un vase d'un fluide pesant à surface libre. Ce problème se ramène au suivant: soit Ω un ouvert de R^n (ou plus généralement d'un espace de Riemann), S sa frontière supposée assez régulière, composée de deux parties disjointes S_0 et S_1 ; chercher les fonctions propres u telles que $\Delta u = 0$ dans Ω , $u = 0$ sur S_0 , $\partial u / \partial n + \lambda \varrho u = 0$ sur S_1 , ϱ désignant une fonction bornée de signe variable définie sur S_1 et λ la valeur propre correspondant à u . Si $S_0 = \emptyset$ et $\int_{S_1} \varrho d\sigma \neq 0$, on impose en outre $\int_{S_1} \varrho u d\sigma = 0$; si $\int_{S_1} \varrho d\sigma = 0$, on ne définit u qu'à une constante additive près. L'auteur substitue au problème précédent celui de la recherche des fonctions et valeurs propres d'un opérateur complètement continu défini dans l'espace d'Hilbert des fonctions nulles sur S_0 en un sens généralisé ou vérifiant les conditions indiquées plus haut si $S_0 = \emptyset$, la métrique étant définie par l'intégrale de Dirichlet. Les démonstrations de complète continuité reposent sur des théorèmes d'extraction dus à W. Kondrachov [C.R. (Dokl.) Acad. Sci. URSS (N.S.) 48 (1945), 535-538; MR 8, 32] et repris par S. Sobolev dans son livre intitulé „Quelques applications de l'analyse fonctionnelle à la physique mathématique” [Izdat. Leningrad. Gos. Univ., 1950; MR 14, 565]; on trouve dans le présent travail des démonstrations simplifiées de ces théorèmes. L'auteur discute également la dérivabilité indéfinie des fonctions propres qu'il obtient, en recourant à un théorème de L. Schwartz [Théorie des distributions, t. I, Hermann, Paris, 1950; MR 12, 31; cf. aussi L. Gårding, Kungl. Fysiogr. Sällsk. i Lund Förh. 20 (1950), 250-253; MR 12, 708].

Une grande partie du travail est consacrée à l'étude de la distribution asymptotique des valeurs propres λ par une généralisation des méthodes de H. Weyl et R. Courant [Courant et Hilbert, Methoden der mathematischen Physik, Bd. I, 2 Aufl., Springer, Berlin, 1931]. Voici le résultat obtenu: si $A^\pm(\lambda)$ désigne resp. le nombre des valeurs propres positives $\geq \lambda$ ou négatives $\leq -\lambda$,

$$A^\pm(\lambda) \sim \frac{\omega_{n-1} \lambda^{n-1}}{(2\pi)^{n-1}} \int_{S_1^\pm} (\pm \varepsilon)^{n-1} d\sigma,$$

où S_1^\pm est la portion de S_1 sur laquelle ϱ est respectivement positif ou négatif et ω_{n-1} le volume de la sphère unité à $n+1$ dimensions. H. G. Garnir (Liège).

Maurin, K. Lösbarkeit der Randwertaufgaben für allgemeine starkelliptische Systeme mit Hilfe des Galerkin'schen Verfahrens. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 207-212.

Mihlin [cf. MR 16, 40, 41] a appliqué la méthode de

Galerkine à l'approximation des solutions des problèmes de Dirichlet, Neumann etc. pour les opérateurs différentiels linéaires du deuxième ordre à coefficients bornés dans un ouvert borné. L'A. traite le cas analogue d'ordre $2m$, $m > 1$, le problème de Dirichlet étant pris comme dans Gårding [Math. Scand. 1 (1953), 55-72; MR 16, 366]; cas du problème de Neumann; la méthode est générale pour tous les problèmes aux limites, pris comme Gårding, Browder ou le Référent. J. L. Lions.

Il'in, A. M. On Dirichlet's problem for an equation of elliptic type degenerating on some set of interior points of a region. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 9-12. (Russian)

The author considers the equation

$$(1) \quad L(u) = a_{11}(x, y) \frac{\partial^2 u}{\partial x^2} + 2a_{12}(x, y) \frac{\partial^2 u}{\partial x \partial y} + a_{22}(x, y) \frac{\partial^2 u}{\partial y^2} + b_1(x, y) \frac{\partial u}{\partial x} + b_2(x, y) \frac{\partial u}{\partial y} + c(x, y)u = h(x, y),$$

where $c(x, y) \leq 0$, $a_{11}(x, y) + a_{22}(x, y) > 0$, and $a_{11}a_{22} - a_{12}^2 \geq 0$ in a bounded closed domain \bar{D} . In the last relation equality is permitted on a set M , consisting of a finite number of interior points Q_i , and a finite number of curves l_i having bounded curvature and intersecting with each other and with the boundary Γ of D in a finite number of points.

It is assumed that there are at most a finite number of points on the curves l_i at which the characteristic direction associated with (1) coincides with the direction of the curve. At the points of $\Gamma \cap M$, it is assumed that a circle of contact can be drawn from the outside such that the direction of the common tangent differs from the characteristic direction at the point. Under these hypotheses and some regularity assumptions, the author establishes the existence of a unique solution of the Dirichlet problem for (1) in D . The proof is based on the use of auxiliary functions and on an estimate of S. Bernstein [C.R. Acad. Sci. Paris 151 (1910), 636-639]. (The reviewer remarks that the estimate as originally published by Bernstein is incorrect. The author refers also to B. A. Barabanov [Dissertation, Saratov Univ., 1939], who presumably has shown the validity of the estimate in the present case.)

Examples that have been given in other connections by the reviewer [Trans. Amer. Math. Soc. 75 (1953), 385-404, esp. p. 399; Ann. of Math. (2) 60 (1954), 397-416, esp. p. 399; MR 15, 430; 16, 592] show that the author's result is best possible in two senses, namely, the assumptions that the curves l_i contain no characteristic segment and that $a_{11} + a_{22} > 0$ in \bar{D} cannot be dropped. It should be noted also that no restriction is made on the order of vanishing of the discriminant on the singular curves and points.

The author indicates several extensions of the result. R. Finn (Los Angeles, Calif.).

Skorobogat'ko, V. Ya. On domains of solvability of Dirichlet's problem for self-adjoint elliptic equations. Ukrain. Mat. Z. 7 (1955), 91-95. (Russian)

Let D be a bounded region with boundary S in n -space with coordinates $x = (x_i)$. Let $c(x)$ and $(a_{ij}(x)) = A$ have two continuous derivatives in D and consider the Dirichlet problem for the elliptic equation $\sum \partial_i (a_{ij} \partial_j u) + cu = 0$. Define $N = \sup_x \sup_i \xi^T A^{-1} \xi$ with $\sum \xi_i^2 = 1$ and put $c^*(x) = c(x)/N$. It is first shown to be sufficient for the solvability of the Dirichlet problem for D that there exist on the closure \bar{D} continuous functions w_i with piecewise-

continuous derivatives $\partial_i w_i$, such that on \bar{D} $\sum w_i^2 + c^* < \sum \partial_i w_i$. Two more explicit criteria are then derived. If $G(x, \xi)$ is Green's function for Laplace's operator and the first boundary-value problem for D , one criterion is that $c^*(x)/D G(x, \xi) d\xi < 1$. Now let $d(x, S)$ denote the distance between x and S and define the intrinsic diameter d of D to be $d = 2 \max_x d(x, S)$. The second criterion is that $d^2 \sup_x c^*(x) < \pi^2$.

F. A. Ficken.

Protter, M. H. Uniqueness theorems for the Tricomi problem. II. J. Rational Mech. Anal. 4 (1955), 721-732.

[For part I see same J. 2 (1953), 107-114; MR 14, 654.]

F. Frankl raised the question of the existence of transonic potential flows out of a jet and showed that such problems when considered in the hodograph plane, led to the Tricomi problem

$$(*) \quad K(\eta) \psi_{\eta\eta} + \psi_{\eta\eta} = 0$$

with $K(\eta)$ a specific monotone function. The author proves the theorem: let D be the specific domain under consideration of the Tricomi problem (enclosed by a simple rectifiable arc Γ lying in the upper half-plane with end-points on the x -axis and the characteristics γ_1, γ_2 passing through these end-points) and suppose $K(\eta)$ has a continuous third derivative satisfying the condition $K'''(\eta) \leq 0$ whenever

$$F(\eta) = \frac{3K''(\eta) - 2K(\eta)K'(\eta)}{K'^2(\eta)} < 0 \quad (\eta < 0).$$

If ψ is a solution of (*) in D which vanishes on Γ and γ_2 the ψ vanishes identically. In a former paper [cf. M. H. Protter, Duke Math. 21 (1954), 1-7; MR 15, 628] the author has given the definition of "normal curves" and has proved the following theorem: let D be the domain described above and suppose Γ coincides with the normal curve of (*) in some neighborhood of the endpoints of γ_1, γ_2 . Then either there exists a unique quasi-regular solution of (*) with prescribed values on Γ and γ_2 or there exist (non-identically vanishing) solutions of (*) which vanish on Γ and γ_2 . This second theorem can be combined with Frankl's uniqueness theorem established for equation (*). It yields an existence theorem (in the hodograph plane) for the transonic flow out of a nozzle. M. Pinl.

Thomée, Vidar. Estimates of the Friedrichs-Lewy type for a hyperbolic equation with three characteristics. Math. Scand. 3 (1955), 115-123.
Soit l'opérateur

$$L = \prod_{i=1}^3 \left(\frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial x_3} \right) + \sum_{i,k=1}^3 a_{ik} \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^3 b_i \frac{\partial}{\partial x_i} + c,$$

où α_1, α_2 et α_3 sont des constantes réelles et a_{ik}, b_i, c des fonctions réelles continues de x définies dans un ouvert borné $V \subset \mathbb{R}^3$ à frontière régulière. La frontière S de V est décomposée en parties selon l'orientation du vecteur normal n par rapport aux régions déterminées par le cône caractéristique de L à l'origine. Le problème étudié consiste à déterminer une solution de $Lu = f$ dans V , connaissant sur S les valeurs de $u, du/dn, \dots, d^p u/dn^p, p \leq 2$ dépendant de la partie de S considérée; on le rencontre dans des questions d'aérodynamique [cf. R. Courant et K. O. Friedrichs, Supersonic flow and shock waves, Interscience, New-York, 1948; MR 10, 637]. Moyennant quelques hypothèses de commodité sur S , l'auteur établit l'unicité de la solution éventuelle et estime l'intégrale dans V de la somme des carrés de ses dérivées d'ordre ≤ 2 en fonction des données sur S et des valeurs de f dans V .

H. G. Garnir (Liège).

Bulah, B. M. Propagation of discontinuities of higher derivatives along characteristics. Uspehi Mat. Nauk 10 (1955), no. 2(64), 143-145. (Russian)

Die Diskussion der Lösungen der quasilinearen partiellen Differentialgleichung zweiter Ordnung

$$L[u] = a(x, y, u, u_x, u_y) u_{xx} + b(\dots) u_{xy} + c(\dots) u_{yy} = d(\dots)$$

führt längs charakteristischer Kurven C der Lösungsflächen im hyperbolischen Gebiet auf Verzweigungsmöglichkeiten, die mit Unstetigkeiten für die zweiten und höheren Ableitungen der Lösungsfunktion $u(x, y)$ verbunden sind. Bezeichnet man den Sprung der Funktion f beim Durchqueren der Charakteristik C mit $\{f\}$, so gilt, wenn die Charakteristik C durch die Beziehung $\varphi(x, y) = 0$ gegeben wird:

$$\{u_{xx}\} \varphi_y - \{u_{xy}\} \varphi_x = 0, \quad \{u_{xy}\} \varphi_y - \{u_{yy}\} \varphi_x = 0,$$

$$\{u_{xx}\} = \kappa \varphi_x^2, \quad \{u_{xy}\} = \kappa \varphi_x \varphi_y, \quad \{u_{yy}\} = \kappa \varphi_y^2,$$

wie bereits von R. Courant und D. Hilbert angegeben worden ist [Methoden der mathematischen Physik, Bd. II, Springer, Berlin, 1937, Kap. V, § 1, S. 297-299]. Dabei kann der Proportionalitätsfaktor κ als ein Maß der Unstetigkeiten der zweiten äußeren Ableitungen angesehen werden. κ ist längs C nicht willkürlich, sondern genügt längs C einer im allgemeinen nichtlinearen Differentialgleichung. Verfasser behandelt insbesondere den Fall, daß diese Differentialgleichung durch eine Bernoullische Differentialgleichung der Form

$$(*) \quad \varphi \kappa(\varphi) \frac{d\kappa}{d\varphi} + \kappa[\beta(\varphi)\kappa + \gamma(\varphi)] = 0$$

gegeben ist und behandelt die weitere Komplikation des Problems im Falle parabolischer Punkte, für welche, zufolge der Beziehung $b^2 - 4ac = 0$ der Koeffizient der Ableitung $d\kappa/d\varphi$ in (*) verschwindet. Dann kommt es auf das Verhalten des Grenzwerts $\lim_{\varphi \rightarrow 0} \kappa$ an. Bernoullische Gleichungen (*) stellen sich z.B. auch in der Theorie der nichtstationären Gasströmungen ein [vgl. R. Sauer, Anfangswertprobleme bei partiellen Differentialgleichungen, Springer, Berlin, 1952, § 27, 2; Ing.-Arch. 18 (1950), 239-241; MR 14, 559; 13, 296; Hantzsche und Wendt, Jbuch. Deutsch. Luftfahrtforschung 1940, 1536-1538; MR 9, 217].

M. Pinl (Köln).

Zautykov, O. A. Cauchy's problem in a linear normed space. Izv. Akad. Nauk Kazah. SSR 1952, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 77-80. (Russian. Kazak summary)

The author (certain misprints in the statement of his result being suitably interpreted) shows an existence and uniqueness theorem for the Cauchy problem:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + xy u + a(x, y) u,$$

$$u(x, y) \Big|_{x=0} = \varphi_1(y), \quad \frac{\partial u}{\partial x} \Big|_{x=0} = \varphi_2(x),$$

where x and y are real variables, and the functions involved have values in a complete, linear, normed vector space.

J. B. Diaz (College Park, Md.).

Teodorescu, N. L'onde de choc dans la théorie invariante de la propagation des ondes. Rev. Math. Phys. 2 (1954), 107-123 (1955).

The author studies the equations satisfied by discontinuities in a function u and its first and second

derivatives where u satisfies the hyperbolic differential equation

$$a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu = f.$$

The function u and its derivatives are assumed to be discontinuous across a characteristic surface of this differential equation. The variation of the jumps in u and the jumps in the derivatives of u along the bicharacteristics are then studied. A. H. Taub (Urbana, Ill.).

Ludford, G. S. S. Generalised Riemann invariants. Pacific J. Math. 5 (1955), 441-450.

The author discusses the Monge-Ampère equation

$$(\ast) \quad r^2 - s^2 + \lambda^2 = 0, \quad \lambda = \lambda(x, y),$$

$\lambda = \lambda(x, y)$ being a simultaneous solution of two nonlinear partial differential equations, one of the second order and the other of the third. In this case $\lambda(x, y)$ must have one of the two forms

$$\lambda = F(\alpha x + \beta y), \quad \lambda = \frac{1}{(x + \alpha)(y + \beta)} F\left(\frac{x + \alpha}{y + \beta}\right)$$

and (\ast) does possess intermediate integrals, or "generalised Riemann invariants". In the fluid-dynamical application of the theory the variables x, y, z, p, q, λ^2 must be replaced by $p, \psi, \xi, t, u, -\tau$, respectively, getting a table of generalised Riemann invariants. The author considers as one example the case of a polytropic gas for which

$$\lambda^2 = \Psi(\psi) p^k, \quad k = \frac{\gamma + 1}{\gamma},$$

where Ψ is a function determined by the (given) distribution function

$$\Psi(\psi) = \frac{A}{(\psi + \beta)^{k+1}}, \quad A, \beta \text{ arbitrary constants.}$$

Any motion for which the relation between τ, p, ψ is of the form

$$p = \frac{A(\psi)}{e} + B(\psi), \quad \tau = \frac{1}{e}; \quad A, B \text{ arbitrary,}$$

possesses generalised Riemann invariants.

M. Pinl (Cologne).

Pini, Bruno. Estensione al caso parabolico di un teorema di F. Riesz relativo alle funzioni subarmoniche. Riv. Mat. Univ. Parma 5 (1954), 269-280.

Set $f(\theta) = [\log(1/\sin^2 \theta)]^k$ and let $C = C(x_0, y_0; r) = C(P_0; r)$ denote the plane curve $x = x_0 + \sqrt{r} f(\theta) \sin \theta$, $y = y_0 + \sqrt{r} f(\theta) \cos \theta$, $-\pi/2 \leq \theta \leq \pi/2$. From $v = v(x, y)$ construct the function

$$\mu(v, P_0, r) = \left(\frac{3}{2\pi}\right)^{1/2} \int_{-\pi/2}^{\pi/2} (v) c f(\theta) \sin^2 \theta \cos \theta d\theta \\ + \frac{1}{r} \left(\frac{2}{3\pi}\right)^{1/2} \int_0^r \int_{-\pi/2}^{\pi/2} v \frac{\sin^2 \theta \cos \theta}{f(\theta)} d\theta dr.$$

A function v is said to be \mathcal{M} -subvalent in a region A if (a) $-\infty \leq v < +\infty$, (b) v is upper semicontinuous in A , and (c) if for each point P of A and for all r sufficiently small $\mu(v, P, r)$ exists and satisfies the inequality $v(P) \leq \mu(v, P, r)$. \mathcal{M} -subvalent functions are related to the operator $\mathcal{M} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in somewhat the same manner as subharmonic functions are related to the operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ [Pini, Ann. Mat. Pura Appl. (4) 37 (1954), 249-264; MR 16, 593]. If the functions $\chi_1(y) <$

$\chi_2(y)$, $a < y \leq b$, are continuous and satisfy Lipschitz conditions, the set of points (x, y) such that $\chi_1(y) < x < \chi_2(y)$, $a < y < b$, will be called a normal domain. The main result of the paper is the following representation theorem. If $v(P)$ is \mathcal{M} -subvalent in a region A , and D is any normal domain contained in A , then there exists a distribution of mass $\omega(e) \geq 0$ such that for P in $Dv(P)$ can be written

$$v(P) = \iint_D V(Q, P) d\omega(e_Q) + h(P).$$

Here $V(P_0, P) = 0$, for $y_0 \leq y$, and $V(P_0, P) = -(y_0 - y)^{-1} \exp(-(x_0 - x)^2/4(y_0 - y))$, for $y_0 > y$. The function $h(P)$ is continuous on the closure of D and in D satisfies the parabolic equation $u_{xx} + u_y = 0$. F. G. Dressel.

Baratta, Maria Antonietta. Sopra un problema non lineare di ripartizione del calore. Riv. Mat. Univ. Parma 5 (1954), 363-371.

Let $U^{(1)}(x, t)$ and $U^{(2)}(x, t)$ represent respectively the temperature in the medium S_1 to the right and the medium S_2 to the left of the plane $x=0$. Assuming S_1 and S_2 have different physical properties, that the film transfer factor of the plane $x=0$ is a function of $\Phi \equiv U^{(1)}(0, t) - U^{(2)}(0, t)$, and also that in this plane there is a source $T = T(t)$ of heat, the author is led to the boundary-value problem

$$a_i^2 U_{xx}^{(i)} = U_t^{(i)}, \quad -(-1)^i x > 0, \quad t > 0 \quad (i=1, 2),$$

$$k_i U_x^{(i)}(0, t) = G(\Phi) - C_i T, \quad t > 0,$$

$$U^{(i)}(x, 0) = 0, \quad -(-1)^i x \geq 0.$$

Here a_i, k_i, C_i are constants, and the known functions $T(t), G(\Phi)$ are bounded and continuous. The function G satisfies a Lipschitz condition and $G(0)=0, G(\Phi) \geq 0$, for $\Phi \geq 0$. Once $\Phi = \Phi(t)$ is known, the solutions $U^{(i)}$ can be written down. In this paper the determination of Φ is reduced to the problem of solving the nonlinear Volterra equation

$$\Phi(t) = \mu(t) + c \int_0^t \frac{G(\Phi(\tau))}{(t-\tau)^{1/2}} d\tau.$$

Here the function $\mu(t)$ is known and c is a constant. [For references to similar problems treated by B. Manfredi, W. R. Mann, J. H. Roberts, and F. Wolf, see Manfredi, same Riv. 3 (1952), 383-396; 4 (1953), 123-132; MR 14, 1090; 15, 228.] F. G. Dressel (Durham, N.C.).

Grosh, R. J., Trabant, E. A., and Hawkins, G. A. Temperature distribution in solids of variable thermal properties heated by moving heat sources. Quart. Appl. Math. 13 (1955), 161-167.

Under the assumption that the thermal conductivity and the product of density and specific heat are linear functions of temperature, the temperature distribution in an infinite rod due to a moving source is found. The same method is applicable to two- and three-dimensional heat-flow problems in infinite solids. It is stated that the method is still applicable if the thermal properties of the solid vary as parabolic or other convenient functions of temperature as long as the thermal diffusivity remains constant. C. G. Maple (Ames, Iowa).

Hantush, M. S., and Jacob, C. E. Non-steady radial flow in an infinite leaky aquifer. Trans. Amer. Geophys. Union 36 (1955), 95-100.

The problem considered is the time variation of the drawdown induced by a well steadily discharging from an

infinite leaky aquifer in which the initial head is uniform. This leads to the boundary-value problem:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} - \frac{s}{B^2} = T \frac{\partial s}{\partial t},$$

$$s(r, 0) = 0, r \geq 0; s(\infty, t) = 0, t \geq 0; \lim_{r \rightarrow \infty} r \frac{\partial s}{\partial r} = -\frac{Q}{2\pi T}, t > 0,$$

where s is the drawdown, r is the radial distance measured from the axis of the well, Q is the discharge of the well and B , S and T are constants of the aquifer.

A solution in the form of an infinite improper integral is obtained by the method of separation of variables and is reduced to the form of an infinite series which converges rapidly for suitably restricted r and t . The variation of drawdown with time at different distances from the well is displayed graphically and in tabular form.

C. G. Maple (Ames, Iowa).

Hantush, M. S., and Jacob, C. E. Non-steady Green's functions for an infinite strip of leaky aquifer. *Trans. Amer. Geophys. Union* 36 (1955), 101-112.

The authors start with the integral form of the solution for drawdown in an infinite leaky aquifer due to a well (see the paper reviewed above) and use the method of images to obtain the drawdown distribution in an infinite strip whose boundaries are either maintained at a constant head or at a vanishing flux. The solution is first obtained in the form of a doubly infinite sum of improper integrals which is reduced to an infinite series by use of a Fourier transform. This latter series is used for computation of the drawdown at the well face, the results of which are displayed in graphical and tabular form. The drawdown distribution is also expressed in terms of a Green's function, and Green's functions for several regions are given.

C. G. Maple (Ames, Iowa).

See also: Vekua, p. 251; Harik, p. 256; Vorović, p. 273; Sauer, p. 296; Lyusternik, p. 303; Berger, p. 319; Šatašvili, p. 320.

Difference Equations, Special Functional Equations

Wright, E. M. A non-linear difference-differential equation. *J. Reine Angew. Math.* 194 (1955), 66-87.

This work culminates a series of notes by the author on difference-differential equations [*Quart. J. Math. Oxford Ser.* 17 (1946), 245-252; *Amer. J. Math.* 70 (1948), 221-238; *Proc. Roy. Soc. Edinburgh. Sect. A.* 62 (1949), 387-393; 63 (1950), 18-26; *MR* 8, 385; 9, 592; 11, 182; 12, 106]. A substantial study is made of the equation

$$(1) \quad y'(x) = -\alpha y(x-1)\{1+y(x)\} \quad (\alpha > 0).$$

The methods used and the results obtained could be extended (but not easily) to equations of a considerably more general type. The case of (1) with $\alpha = \log 2$ occurs in the application of probability methods to the distribution of prime numbers. Some of the results are as follows. If $\alpha \leq \frac{1}{2}$ and $y(0) > -1$, then $y \rightarrow 0$ for $x \rightarrow \infty$; if $\alpha > \frac{1}{2}$, solutions y exist, with $y(0) > -1$, not $\rightarrow 0$ with $1/x$. The linear equation $z'(x) = -\alpha z(x-1)$ has a solution of form $\sum A_n e^{s_n x}$, where the sum is over the zeros s of $se^s + \alpha$; these zeros are studied in considerable detail. If for $x \rightarrow \infty$ one has $y = O(e^{-Cx})$ ($C > 0$), then y has an asymptotic representation in terms of exponential functions, with coefficients polynomial in x . A number of theorems

relates to solutions of (1) in the form of absolutely and uniformly convergent series of exponential functions. Suppose y is a solution of (1) for all real x ; if $y \rightarrow -1$ (for $x \rightarrow -\infty$), then y is one of

$$-1, y_1(x-x_0) = -1 - \sum_1^\infty c_n e^{n\alpha(x-x_0)},$$

$$y_2(x-x_0) = -1 - \sum_1^\infty (-1)^n c_n e^{n\alpha x}$$

(the c_n are certain constants); for any other solution one has $-1 < y < e^2 - 1$. If y (solution for all real x) $\rightarrow 0$ for $x \rightarrow -\infty$ and $\alpha < \frac{1}{2}\pi$, then $y = 0$; if $\frac{1}{2}\pi < \alpha < 5\pi/2$, then $y = 0$ or y is a certain series in exponentials of $(x-x_0)$. If $y(x)$ is a solution for all real x , $y(x)$ is analytic in a strip containing the axis of reals.

W. J. Trjitzinsky (Urbana, Ill.).

Aczél, J. Lösung der Vektor-Funktionalgleichung der homogenen und inhomogenen n -dimensionalen einparametrischen „Translation“, der erzeugenden Funktion von Kettenreaktionen und des stationären und nichtstationären Bewegungsintegrals. *Acta Math. Acad. Sci. Hungar.* 6 (1955), 131-141. (Russian summary)

The author considers the vector functional equations (1) $f[f(x, t), u] = f(x, t+u)$ and (2) $f[f(x, s, t), t, u] = f(x, s, u)$, where x is a vector $\{x_1, \dots, x_n\}$ and s, t, u are scalars. The principal results: I. Let $h(x)$ be an arbitrary vector function possessing a unique inverse $g: h[g(y)] = y$, and define $y+t = \{y_1, \dots, y_n\} + t$ to be the vector $\{y_1+t, y_2, \dots, y_n\}$. Then (3) $f(x, t) = g[h(x)+t]$ is a solution of (1). Moreover, conversely, given that f satisfies (1) on the hyperplane $x = x_0 = \{a, x_2, \dots, x_n\}$ where a is constant, and that for every x there is an x_0 such that $x = h(x_0)$ such that $g(x_0) = f(x_0, t) = x$, then f is given by (3), so (1) holds for all x, t, u . II. Let $y = h(x, t)$ be an arbitrary vector function with parameter t , having an inverse $x = g(y, t)$; $h[g(y, t), t] = y$. Then the function (4) $f(x, t, u) = g[h(x, t), u]$ satisfies (2). And if f satisfies (2) on the hyperplane $x = x_0$, and for every x and t there is an $x_0 = \{s, x_2, \dots, x_n\} = h(x_0, t)$ such that $g(x_0, t) = f(x_0, s, t) = x$, then f is given by (4) and therefore satisfies (2) for all x, s, t, u . Some applications are considered.

I. M. Sheffer.

See also: Hosszú, p. 236.

Integral Equations, Equations in Infinitely Many Variables

Bellman, Richard. Dynamic programming and a new formalism in the theory of integral equations. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 31-34.

This paper provides an application of dynamic programming to obtain a functional equation for the characteristic values of the integral equation

$$\lambda u(t) = \int_0^T K(s, t) u(s) ds.$$

T. L. Saaty (Washington, D.C.).

Bellman, Richard, Glicksberg, Irving, and Gross, Oliver. On some nonlinear integral equations occurring in the theory of dynamic programming. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 227-229.

This paper gives a summary of an application of dynamic programming to finding solutions of a nonlinear integral equation connected with the problem of optimal inventory.

T. L. Saaty (Washington, D.C.).

Pekeris, C. L. Solution of the Boltzmann-Hilbert integral equation. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 661-669.

Der Verfasser behandelt die lineare Integralgleichung, die Hilbert bei seinem Ansatz zur Lösung der Maxwell-Boltzmann'schen Integrodifferentialgleichung durch eine asymptotische Reihe, deren Konvergenz noch in keinem Fall bewiesen wurde, aufgestellt hat. Aus dieser werden üblicherweise Näherungsausdrücke für Zähigkeit, Wärmeleitfähigkeit und Selbstdiffusion gewonnen. Um die dabei auftretende „orthogonale“ [E. Hecke, Math. Ann. 78 (1918), 398-404; Math. Z. 12 (1922), 273-286] zu lösen, entwickelt der Verfasser deren Kern nach räumlichen Kugelfunktionen und führt so die Berechnung der physikalischen Größen auf die Lösung gewöhnlicher Differentialgleichungen zurück, die, wie er am Schluß bemerkt, sich auch bei Boltzmann finden lassen. Die zahlenmäßigen Ergebnisse werden mit denen von Chapman und Cowling [The mathematical theory of non-uniform gases, 2nd ed., Cambridge, 1952] bzw. Pidduck [Proc. London Math. Soc. (2) 15 (1916), 89-127] verglichen.

D. Morgenstern (Berlin).

Rogosinski, W. W. On finite systems of linear equations, with an infinity of unknowns. Math. Z. 63 (1955), 97-108.

The finite system of equations to be solved is $\sum_k a_{ik}x_k = C_i$ ($i=1, \dots, n$) under the assumptions (a) a_{ik} in l^p for each i , solution x_k in $l^{p'}$ ($\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$); (b) a_{ik} in l^1 for each i , x_k in m (bounded sequences); (c) a_{ik} in c with $\lim_k a_{ik} = a_{i0}$ for each i , x_k in l^1 . It is well known that solutions minimizing the obvious norm exist in each of these cases and Riesz [Les systèmes d'équations linéaires à une infinité d'inconnues, Gauthier-Villars, Paris, 1913, pp. 49ff.] has shown that these solutions are unique in case (a). For case (b) the paper proves that there exists at least one index K such that x_K is the same for all minimal solutions and $\|x_K\| = \|x\|$. For the case (c) every minimal solution is directed in the sense that either the x_k ultimately vanish or $\arg x_k \rightarrow \varphi \pmod{2\pi}$ for those k for which $x_k \neq 0$ and that all such solutions are equally directed in that $\arg x_k' = \arg x_k''$ when both $x_k' \neq 0$ and $x_k'' \neq 0$ and the φ is the same for both. In case $a_{i0} = 0$ for all i , there exists a K the same for all minimal solutions such that $x_k = 0$ for $k > K$. To prove these theorems, the problem is given the general setting [cf. Banach, Théorie des opérations linéaires, Warsaw, 1932, pp. 76ff]: Given the finite system of linear functional equations $B(a_i) = C_i$ ($i=1, \dots, n$), where a_i are points in a linear normed space \mathfrak{A} , and the linear form B is sought. Then B is determined on the linear space \mathfrak{P}_n spanned by the a_i and has a norm $\|B\|$ on \mathfrak{P}_n . The Hahn-Banach extension theorem gives the extension of B to all of \mathfrak{A} with preservation of norm, i.e., the B^* of minimal norm on \mathfrak{A} satisfying the given equations, as well as its properties. Extension to an infinite number of equations or replacement of \mathfrak{P}_n by a linear subspace of \mathfrak{A} follows, leading to detailed consideration of the original special cases where \mathfrak{A} is l^p , l^1 or c .

T. H. Hildebrandt.

Vorovič, I. I. On the existence of solutions in the non-linear theory of shells. Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 173-186. (Russian)

Le problème est ramené à la résolution de

$$w(P) = \int_G G(P, Q) A(w) dQ,$$

$P, Q \in C$ = ouvert borné du plan; G est la fonction de

Green du problème de Dirichlet dans C pour Δ^2 (soient φ , et λ , les fonctions propres et valeurs propres de ce problème); $A(w)$ est un opérateur différentiel non linéaire du deuxième ordre, compliqué. L'A. montre l'existence de solutions; pour cela, il approche G par $G_n(P, Q) = \sum_{i=1}^n \lambda_i^{-1} \varphi_i(P) \varphi_i(Q)$. On cherche w_n solution du système correspondant, w_n étant pris dans l'espace $[\varphi_1, \varphi_2, \dots, \varphi_n]$. L'A. montre que les w_n demeurent dans un compact d'un espace de Hilbert convenable (adhérence des fonctions à support compact dans $W^2(C)$, $W^2(C)$ étant l'espace des fonctions dans $L^2(C)$ ainsi que leurs dérivées des deux premiers ordres); toute limite des w_n est solution du problème.

J. L. Lions (Nancy).

See also: Allen, p. 283.

Theory of Probability

Bochner, Salomon. Harmonic analysis and the theory of probability. University of California Press, Berkeley and Los Angeles, 1955. viii+176 pp. \$4.50.

In this book, the author elaborates some of his work of the last decade, in which he has formulated a very general concept of stochastic process, and has applied to it various modern techniques of harmonic analysis. The probabilistic interpretation of the concepts introduced is frequently omitted. Moreover, the formalistic approach does not make for easy reading, and a reader not familiar with the ideas may have some difficulty separating the technical trivia from the important concepts and results. Applications are hinted at, but not given in detail. The following outline indicates the topics treated.

Chapters 1 and 2, "Approximations" and "Fourier expansions", include the necessary introductory material on approximation in various senses, and on Fourier expansions, stressing the use of general convergence factors. In this way, the connections between expansion problems and certain boundary-value problems for Laplace's equation and the heat equation are established. Chapter 3, "Closure properties of Fourier transforms", contains a general analysis of the principal closure properties of families of characteristic functions and of related families. The family of characteristic functions of the random variables of a stochastic process with stationary independent increments (infinitely subdivisible process) is treated, and Lévy's formula for the characteristic function of an infinitely divisible distribution is obtained. The treatment is more general than the usual one in that the k -dimensional case (also covered by Lévy) is treated and that the basic kernels $e^{i(\alpha, x)}$, $1 - \cos(\alpha, x)$ are replaced in much of the discussion by more general "pseudo-characters" and "Poisson characters". The treatment is based on that in a paper by the author [Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 1082-1088; MR 15, 295]. For example, if B is a Bohr almost periodic function for which 0 is not a characteristic frequency, $B(\alpha, x)$ defines a pseudo-character. In Chapter 4, "Laplace and Mellin transforms", the theory of completely monotone functions on k -space is applied to the study of the subordination of infinitely subdivisible and more general Markov processes. The concept of subordination is due to the author [Duke Math. J. 3 (1937), 726-728], and merits more discussion and explanation than it receives. Chapter 5, "Stochastic processes and characteristic functionals", is based on a 1947 paper [Ann. of Math. (2) 48 (1947),

1014-1061; MR 9, 193] and later papers. The essential idea is that a stochastic process is a randomized set function, but this original concept has been considerably generalized. The characteristic functional is introduced and leads to a dual parameter space. The length of a random path in a homogeneous space is discussed. Chapter 6, "Analysis of stochastic processes", stresses the role of stochastic integrals, leading to expansions in series and integrals.

J. L. Doob (Urbana, Ill.).

Roberts, J. D. A theory of biased dice. *Eureka* no. 18 (1955), 8-11.

Hsu, Pao-Lu. On characteristic functions which coincide in the neighborhood of zero. *Acta Math. Sinica* 4 (1954), 21-32. (Chinese. English summary)

A characteristic function (c.f.) is said to belong to (\bar{U}) if and only if there exists a distinct c.f. to which it is equal in a neighborhood of zero. If $g(t)$ is a c.f. which is even, convex, non-negative and non-increasing on the positive t -axis then it belongs to (\bar{U}) because $\max(g(t), g(a))$ for each $a > 0$ is another c.f. by a well-known criterion. According to the author, all known examples of c.f.'s belonging to (\bar{U}) are subsumed by this simple observation. [But what about Gnedenko's example [see, e.g., Loève, *Probability theory*, Van Nostrand, New York, 1955, p. 218; MK 16, 598]?' Oddly enough the superfluity of the condition $g(t) \rightarrow 0$ as $t \rightarrow \infty$ was noticed neither by Titchmarsh [Introduction to the theory of Fourier integrals, Oxford, 1937, p. 170] nor Pólya [Proc. Berkeley Symposium Math. Statist. Probability, 1945, 1946, Univ. of California Press, 1949, pp. 115-123, p. 116; MR 10, 463] nor Loève.] The stable c.f. $\phi(t) = \exp\{-(1-ic \operatorname{sgn} t)|t|^\alpha\}$ belongs to (\bar{U}) if $0 < \alpha < 1$ and $c \leq b(\alpha)$. This is proved by showing that the function which equals $\phi(t)$ if $|t| < 1$, $\exp\{ic(2-|t|)\operatorname{sgn} t-1\}$ if $1 \leq |t| < 2$, and e^{-1} if $|t| \geq 2$ is a c.f. If $\int_{-\infty}^{\infty} |q(x)| dx = 1$, $q(-x) = \bar{q}(x)$ for all x , such that the Fourier transform of q vanishes in an interval, then the c.f. of the density function (d.f.) $|q(x)|$ belongs to (\bar{U}) . This criterion leads in particular to $(1+x^2)^{-\lambda}$, $\lambda > 1$; $|\sin x/x|^n$; and a class of d.f.'s including

$$\text{const} \times \exp\{-|x| \min(1, 1/\psi(|x|))\},$$

where $\psi(x)$ is any member of the hierarchy $(\ln x)^\lambda$, $(\ln \ln x)^\lambda$, \dots , $\lambda > 1$. Finally it is shown that f belongs to (\bar{U}) if and only if any sequence of c.f.'s which converge to f in a neighborhood of zero must converge to it everywhere. From this it is shown that Zygmund's result relating to this question [Proc. 2nd Berkeley Symposium Math. Statist. Probability, 1950, Univ. of California Press, 1951, pp. 369-372; MK 13, 362] is a consequence of a result by Marcinkiewicz [Fund. Math. 31 (1938), 86-102].

K. L. Chung (Syracuse, N.Y.).

Hsu, P. L. Absolute moments and characteristic functions. *J. Chinese Math. Soc. (N.S.)* 1 (1951), 257-280. (Chinese summary)

Let F be a distribution function and f its characteristic function. If $\beta > 0$, n is a non-negative integer, and if either $2n - \beta > 0$ or $F(x)$ is continuous at $x=0$, then

$$\begin{aligned} \int_{-\infty}^{\infty} |x|^{2n-\beta} dF(x) \\ = \frac{(-1)^n 2^{1-\beta}}{\pi^{1/2} \Gamma(\beta/2)} \int_0^{\infty} y^{-2n+\beta-1} dy \int_{-\infty}^{\infty} e^{-y^2 t^2} H_{2n}(t) f(yt) dt, \end{aligned}$$

where $H_n(t) = (-1)^n e^{t^2} d^n e^{-t^2} / dt^n$. Using this main formula,

it is proved that if k is non-negative integer and $0 < \gamma < 1$, the absolute moment of order $k+\gamma$ of F is finite if and only if in a neighborhood of the origin $f(t) = Q_k(t) + O(|t|^{k+\gamma} \psi(t))$ where $Q_k(t)$ is a polynomial of degree k and $\psi(t)$ is non-negative, bounded such that $|t|^{-1} \psi(t)$ is summable in the neighborhood. An explicit expression of the moment is given in terms of the real part of f . Similar conditions are proved for absolute moments of integral order and for the generalized moment $\int_{|x|>0} |x|^\beta \log |x| dF(x)$. Previous results by Fortet [Bull. Sci. Math. (2) 68 (1944), 117-131; MR 7, 62] are included as special cases. Absolute moments of negative order and the related Stieltjes are transform briefly discussed.

K. L. Chung.

Oshio, Shigeru. On mean values and geometrical probabilities in E_n . *Sci. Rep. Kanazawa Univ.* 3 (1955), no. 1, 35-43.

Let the euclidean space E_n be covered by a uniform system of congruent n -dimensional cells of volume C and let us assume that each cell contains a p -dimensional variety of finite p -dimensional measure V_p ($p < n$); they form a uniform net N_p of p -dimensional varieties. If a q -dimensional variety K_q of finite q -dimensional measure V_q ($q \geq n$) moves in E_n , the mean value of the measure of the $(p+q-n)$ -dimensional intersection variety $K_q \cap N_p$ is given by

$$O_n O_{p+q-n} (O_p O_q C)^{-1} V_p V_q, \text{ where } O_i = 2\pi^{(i+1)/2} / \Gamma((i+1)/2).$$

The author considers the cases $n=3$ and $p=1$, $q=2$; $p=2$, $q=1$; $p=2$, $q=2$ and makes some applications to problems of geometrical probabilities. [The right-hand side of the equation (18) must be replaced by $2\pi^3 F_0 F_1$; see Blaschke, *Vorlesungen über Integralgeometrie*, H. II, Teubner, Leipzig-Berlin, 1937, p. 113.]

L. A. Santaló.

Bellman, Richard. A note on the mean value of random determinants. *Quart. Appl. Math.* 13 (1955), 322-324.

In this note, the moments of determinants with independent random elements are expressed in terms of the characteristic functions of the random elements by use of differential operators which are themselves determinants. Since the random elements need to have the same distribution, this is formally a generalization of the work of Nyquist, Rice and Riordan [Quart. Appl. Math. 12 (1954), 97-104; MR 16, 148]. For the determinant D_n of order n , if $E(D_n^r)$ is the r th moment (about the origin), $\phi_{ij}(z)$ the characteristic function of the element in row i and column j , and θ_n is a determinant of order n with typical element the partial differential operator $\partial/\partial z_{ij}$, the central result is written

$$i^r E(D_n^r) = \theta_n^r [\phi_{11}(z_{11}) \phi_{12}(z_{12}) \cdots \phi_{nn}(z_{nn})]$$

with i the imaginary unit and all (n^2) z 's set at zero after completion of operations. Further development in a subsequent paper is promised.

J. Riordan.

Chao, Chung-Jeh. Explicit formula for the stable law of distribution. *Acta Math. Sinica* 3 (1953), 177-185. (Chinese. English summary)

Expansions of the Fourier transform of

$$\exp\{-(c_0 - ic_1 \operatorname{sgn} t)|t|^\alpha\}$$

with $1 \neq \alpha > 0$, $c_0 > 0$ without assuming that it is a density function. In the latter case the results were obtained by H. Bergström [Ark. Mat. 2 (1952), 375-378; MR 16, 377]. The author acknowledges this and states that he completed his work before seeing Bergström's. K. L. Chung.

Hoeffding, Wassily, and Shrikhande, S. S. Bounds for the distribution function of a sum of independent, identically distributed random variables. *Ann. Math. Statist.* 26 (1955), 439-449.

The problem is considered of obtaining bounds for the (cumulative) distribution function of the sum of n independent, identically distributed random variables with k prescribed moments and given range. For $n=2$ it is shown that the best bounds are attained or arbitrarily closely approached with discrete random variables which take on at most $2k+2$ values. For non-negative random variables with given mean, explicit bounds are obtained when $n=2$; for arbitrary values of n , bounds are given which are asymptotically best in the "tail" of the distribution. (From the authors' summary.) *D. M. Sandelius.*

Cheng, Tseng-Tung. On asymptotic expansions connected with the sums of independent random variables. *Acta Math. Sinica* 5 (1955), 91-108. (Chinese. English summary)

These concern the so-called "large deviations" investigated by Cramér and Feller [Feller, *Trans. Amer. Math. Soc.* 54 (1943), 361-372; MR 5, 125]. Both distribution and density functions are treated. The results are too complicated to be given here. *K. L. Chung.*

Arfwedson, G. Research in collective risk theory. I. *Skand. Aktuarietidskr.* 37 (1954), 191-223 (1955).

This is a continuation of the author's earlier studies [Skand. Aktuarietidskr. 36 (1953), 1-15; MR 15, 238]. He surveys the collective risk theory and introduces the probability $F(x, u)$ that the risk fund of initial value u does not become negative at any time during the period x . Here x is a "transformed time" and represents the expected number of claims. In this first part of paper the functions $F(\infty, 0)$, $F(x, 0)$ and $\psi(u) = 1 - F(\infty, u)$ are studied, mostly by means of Fourier transforms.

E. Lukacs (Washington, D.C.).

Wang, Shou-Jen. On the limiting distribution of the ratio of two empirical distributions. *Acta Math. Sinica* 5 (1955), 253-267. (Chinese. English summary)

Let $S_m(x)$ and $T_n(x)$ be the empirical distributions based on m and n independent observations of a random variable with a continuous distribution function. Let $N = mn/(m+n)$ and $m \rightarrow \infty$, $n \rightarrow \infty$ so that $m/n \rightarrow d \leq 1$. Then for each $a > 0$,

$$\lim_{n \rightarrow \infty} P \left\{ \sup_{0 \leq t \leq T_n(a)} \frac{S_m(x) - T_n(x)}{T_n(x)} < zN^{-1} \right\} = \begin{cases} \sqrt{z/\pi} \int_0^{\sqrt{az/(1-a)}} e^{-t^2} dt, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

This is an analogue of a theorem of Rényi [Acta Math. Acad. Sci. Hungar. 4 (1953), 191-231; MR 15, 885], of which another proof is given. The method is that of Feller's [Ann. Math. Statist. 19 (1948), 177-189; MR 9, 599; 10, 855]. *K. L. Chung (Syracuse, N.Y.).*

Kaplan, Edward L. Transformations of stationary random sequences. *Math. Scand.* 3 (1955), 127-149.

Let $\{v_n, -\infty < n < \infty\}$ be a sequence of random variables. It is supposed that, if $v_n - v_{n-1} = u_n$, the u_n sequence is a stationary sequence of non-negative random variables for which the probability is 0 that all the random variables

vanish simultaneously. The following results are obtained.

(1) Almost no sample v_n sequence has a finite cluster point, and, taking $v_0=0$, the expected number of v_n points in any interval I is at most the corresponding number for the interval I' of twice the length, centered at the origin. A side result is that, if x, y are mutually independent and have a common distribution, the probability that a value of $x-y$ lies in I is less than or equal to the corresponding probability for I' . (2) The v_n sequence yields a stationary flow of events, if v_0 is chosen properly. The reviewer [Trans. Amer. Math. Soc. 63 (1948), 422-438; MR 9, 598] treated the special case in which the u_i are mutually independent. (3) If $\{\mu_n, -\infty < n < \infty\}$ is a stationary sequence of positive integer-valued random variables, with finite expectations, and if, for every integer μ , H_μ is a μ -variate distribution, let

$$\{u_i\} = \dots, w_1^{(0)}, \dots, w_{\mu_1}^{(0)}, w_1^{(1)}, \dots, w_{\mu_1}^{(1)}, \dots$$

be a sequence of random variables such that, for given subscript values, the distribution of $w_1^{(0)}, \dots, w_{\mu_1}^{(0)}$ is H_{μ_1} , and that this set of random variables is independent of all other such sets. Then the u_i sequence is stationary, if u_0 is chosen as $w_m^{(0)}$, where m is a certain random integer. (4) Going back to (1), let the x_i sequence be such that the (u_i, x_i) sequence is stationary, and let $V_i = v_i + x_i$. Then the V_i points almost never have a finite cluster point, and if they are renumbered in increasing order, obtaining $V'_i = V_{\pi(i)}$, with $\Pi(0)=0$, the V'_i sequence has the same properties as the original v_n sequence. (5) Let $k_+(j)$ be the number of points (right crossovers) i with $i > j$, $\Pi(i) < \Pi(j)$, and let $k_-(j)$ be the number of left crossovers. These random variables have equal, possibly infinite, expectations, and each vanishes with positive probability unless it is infinite with probability 1. Conditions for almost certain finiteness, and on the character of the permutation $j \rightarrow \Pi(j)$ in this case, are discussed.

J. L. Doob (Urbana, Ill.).

Gillis, J. Correlated random walk. *Proc. Cambridge Philos. Soc.* 51 (1955), 639-651.

The author considers a random walk on a d -dimensional rectangular lattice. The initial point is the origin, and the first step is a unit step in any one of the axis directions, with common probability $1/(2d)$ for each. At any later stage, the step is a unit step in any one of the axis directions, with probability p for a continuation of the preceding direction, probability q for a reversal of direction, probability r for any other direction. The case $d=1$ has been treated by Goldstein [Quart. J. Mech. Appl. Math. 4 (1951), 129-156; MR 13, 960] and Klein [Proc. Roy. Soc. Edinburgh. Sect. A. 63 (1952), 268-279; MR 14, 295]. Let $\delta = p - q$ and let $P_n(m)$ be the probability that the walk will be at point m in n steps. The author finds the generating function of $P_n(m)$. When $d=1$, he finds the mean and variance of the position, and proves asymptotic normality. When $p=r$, set $P_n(0) = R_{n,d}(\delta)$. It is shown that, if $p=r$, and if $d=1$ or if d is even, then

$$R_{2n,d}(\delta) \sim R_{2n,d}(0) \left(\frac{1-\delta}{1+\delta} \right)^{n/2}$$

J. L. Doob (Urbana, Ill.).

Lévy, Paul. Propriétés asymptotiques de la courbe du mouvement brownien à N dimensions. *C. R. Acad. Sci. Paris* 241 (1955), 689-690.

The following theorems on the Brownian motion process $\{X(t), 0 \leq t < \infty\}$, $X(0)=0$, in N dimensions are

stated. Let Γ_l be the trajectory arc corresponding to the parameter interval $[0, l]$. In the following, l can be either 0 or ∞ . (1) Let $L_p(l)$ be the maximum length of a polygonal line, with p sides, inscribed in Γ_l , with vertices ordered according to the corresponding parameter values. Then

$$\Pr \left\{ \limsup_{l \rightarrow \infty} \frac{L_p(l)}{2l \log \log l} = 1 \right\} = 1.$$

(2) Let C be any continuous image in N -space of a compact line segment, and let C have the origin as initial point. Let $x_i(\tau) = X(\tau)/\lambda(\tau)$, and let γ_i be the locus of $x_i(\tau)$, for $0 \leq \tau \leq l$. Here $\lambda(\tau)$ is chosen to make equal the longest chords of C and γ_i . Then there are almost certainly sequences $\{l_i\}$ such that $l_i \rightarrow l$ and that γ_{l_i} converges to C in the sense of Fréchet distance. For such a sequence, let

$$k = \limsup_{p \rightarrow \infty} \frac{\lambda(t_p)}{\sqrt{(2l_p \log \log t_p)}}.$$

Then, if C is of unit length, $k \leq 1$ and the t_i 's can be chosen to make $\gamma_{l_i} \rightarrow C$, and $k=1$. (3) Let $N=2$, and let $m_1(C)$, $m_2(C)$ be respectively the measure of the smallest convex set containing C and the sum of the measures of the bounded components of the complement of C . Then the preceding results are applied to derive the equation

$$\Pr \left\{ \limsup_{l \rightarrow \infty} \frac{\pi m_1(\Gamma_l)}{l \log \log l} = 1 \right\} = 1 \quad (i=1, 2).$$

The first and third results can be extended, when $l=\infty$, to cases, such as N -dimensional random walks, in which the basic stochastic process has independent increments which are asymptotically isotropic and Gaussian, including those with integer-valued parameters. *J. L. Doob.*

Edwards, D. A., and Moyal, J. E. Stochastic differential equations. *Proc. Cambridge Philos. Soc.* **51** (1955), 663-677.

Let $z(\cdot)$ be a function defined on $[0, T]$, with range space a weakly complete Banach space, and suppose that this function is strongly continuous and of (weak) bounded variation. Let $p(\cdot)$ and $q(\cdot)$ be complex-valued continuous functions on $[0, T]$. The authors show that then the differential equation

$$(1) \quad \frac{d[\dot{x}(t) - z(t)]}{dt} + p(t)\dot{x}(t) + q(t)x(t) = \theta$$

can be solved just as in the numerical case. (All derivatives are strong derivatives.) This result is applied, with two-point boundary conditions, to the case when $\{z(t), 0 \leq t \leq T\}$ is a stochastic process, the Brownian motion process, with the L_2 norm, so that the differential equation is that of a linear system with a Brownian driving force. This type of application, in a special case, goes back to the reviewer [*Ann. of Math.* (2) **43** (1942), 351-369; *MR* **4**, 17] who used a slightly different definition of the solution of (1). *J. L. Doob* (Urbana, Ill.).

Takács, Lajos. On secondary stochastic processes derived by means of recurrence processes. *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **5** (1955), 187-197. (Hungarian)

This is a continuation of the authors earlier work on secondary processes generated by a primary process with independently and identically distributed increments [same *Közl.* **4** (1954), 473-504, 571-587; *MR* **16**, 723]. In the present paper the form of the function $f(u, x)$ is not specified. The author studies in this general set up the secondary process $Y(t)$, the corresponding stationary process, the distribution function $F(t, x)$ of $Y(t)$ and the

corresponding limiting distribution $\lim_{t \rightarrow \infty} F(t, x)$. (For the notation, as well as for the formulation of the mathematical model, see the reviews of the papers cited above.)

E. Lukacs (Washington D.C.).

Urbanik, K. On a stochastic model of a cascade. *Bull. Acad. Polon. Sci. Cl. III.* **3** (1955), 349-351.

The usual description of a cascade or branching process is in terms of the total number of particles, and the histories of individual particles are not part of the mathematical model. Following a suggestion of E. Marczewski, the author sets up a model for a temporally homogeneous cascade process where individual lines of descent can be distinguished. Two theorems are stated without proof.

(1) Every cascade process defined in extensive form determines a temporally homogeneous Markov process whose probabilities satisfy the usual relations for cascade processes. (2) Every Markov process of the type mentioned can be generated by an appropriate extensive cascade process.

T. E. Harris (Santa Monica, Calif.).

Waugh, W. A. O'N. An age-dependent birth and death process. *Biometrika* **42** (1955), 291-306.

An object has a distribution of life-length $G(t)$; if it dies at age t , it is replaced by r particles with probability $q_r(t)$ ($r=0, 1, 2, \dots$), and so on, each particle acting independently of the others. The case $q_0(t) + q_2(t) = 1$ is treated in detail, generalizing the case $q_2 = 1$ treated by R. Bellman and the reviewer [*Proc. Nat. Acad. Sci. U.S.A.* **34** (1948), 601-604; *Ann. of Math.* (2) **55** (1952), 280-295; *MR* **10**, 311; **13**, 664]. The generating function for the population size satisfies a nonlinear integral equation and the factorial moments satisfy renewal-type integral equations. The probability of extinction is, under suitable assumptions, $\min(q/\sigma, 1)$ where $q = \int_0^\infty q_0(u) dG(u)$, $\sigma = \int_0^\infty q_2(u) dG(u)$. Let $B(t)$ and $D(t)$ be the life-length distributions for an individual which would obtain, respectively, if the risk of dying without issue were suspended or if the risk of dying with the creation of two new particles were suspended. A detailed treatment is given of the case where $B'(t) = (k\lambda)^{k-1} t^{k-1} e^{-k\lambda t} (k-1)!$ and

$$D'(t) = (mv)^m t^{m-1} e^{-mv t} / (m-1)!,$$

a more general form of a model studied by D. G. Kendall [*Biometrika* **35** (1948), 316-330; *MR* **10**, 385]. Asymptotic behavior for large t is studied and in the case $m=1$ explicit formulas are obtained for the first and second moments. Much of the discussion is in terms of the life-length distributions of (a) those individuals terminating without issue and (b) those individuals terminating with creation of two new individuals. *T. E. Harris.*

★ **Hinčin, A. Ya.** Matematičeskije metody teorii massovogo obsluživaniya. [Mathematical methods of the theory of mass service.] *Trudy Mat. Inst. Steklov.* v. 49. Izdat. Akad. Nauk SSSR, Moscow, 1955. 122 pp. 5.75 rubles.

The author presents a treatment of his subject written in an elementary style, accessible to readers expert neither in probability nor in the technical background of the subject. The treatment is not and does not intend to be complete, but was written to acquaint the readers with the basic ideas and methodology of the subject. The treatment is thus centered around the classical work of Erlang [Brockmeyer, Halstrøm, Jensen, Trans. Danish Acad. Tech. Sci. **1948**, no. 2; *MR* **10**, 385] and the more recent work of Palm [Ericsson Technics no. **44** (1943);

MR 6, 160]. The author has, however, completed and simplified their work in some respects.

The book is divided into three parts. The first part gives a detailed discussion of the idea of a stationary stream of events, "calls", with most stress on the simplest case (Poisson process) when the numbers of calls in non-overlapping intervals are independent, and on the slightly more general case when the intervals between calls are independent random variables with a common distribution function.

The second part is devoted to systems with losses. The incoming calls are Poisson, there are n ($n \leq \infty$) servers, and if there are no free servers the call is lost. The service times have exponential distributions. The number of servers busy at time t then determines a Markov chain, and this chain is treated in detail, under each of the following two hypotheses. Under the first hypothesis, the server order is irrelevant. Under the second hypothesis, the servers are ordered, and each call is served by the lowest numbered free server.

The third part deals with systems having waiting times, that is, each call is held until a server is available, and the waiting time distribution is investigated. The first two chapters assume that the calls are Poisson, and that the service time is, respectively, exponential or identically constant. The last chapter, based on a paper by the author [Mat. Sb. 39 (1932), no. 4, 73-84], assumes that there is only one server, that the calls are Poisson, and that the service time has an arbitrary distribution except that the expectation is finite.

J. L. Doob.

Cox, D. R. The statistical analysis of congestion. J. Roy. Statist. Soc. Ser. A. 118 (1955), 324-335.

This is an expository paper mainly directed to workers in operational research. Ideas are illustrated by (1) the single-server problem with general service time and random or regular arrivals, and (2) machine-minding by a single operative, distinguished from (1) by the limited number of machines and hence of load on the server. These examples are discussed carefully and in detail with attention to the way they represent practical situations and to the way theoretical results may be used; it is noticed, for instance, in example (1), that a considerable reduction in mean waiting time appears when arrivals are regular rather than random, or again that an "occupancy" of unity, which tends to fully occupy the server and is certain to produce an infinite waiting line given sufficient time, need not be ruled out for this reason if service times are not too widely dispersed and the time of operation of interest is short. Nonhomogeneous traffic and traffic with priorities are also discussed. Finally the connection of traffic with inventory and storage problems is mentioned and an appendix gives some approximate results for the study of traffic flow in time, that is, for the rate of growth of a traffic build-up and for the time of evanescence of a long waiting line.

J. Riordan (New York, N.Y.).

Schützenberger, P. Les problèmes de diagnostic et l'axiomatique des informations. Rev. Gén. Sci. Pures Appl. 62 (1955), 222-226.

The two distinct definitions of (expected) amount of information, due to R. A. Fisher and to C. E. Shannon and N. Wiener, are subsumed under a more general form. It is claimed that this form arises naturally by thinking of amount of information as related to the cost of experimentation and to decisions to be taken after an experiment. Detailed references and demonstrations are omitted.

I. J. Good (Cheltenham).

See also: van Dantzig, p. 227; Berge, p. 228; Bahadur, 286; Linfoot, p. 322.

Mathematical Statistics

Jordan, Károly. On some new results in probability calculus. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 129-135. (Hungarian)

The paper consists of two parts, each containing a lecture recently given by the author. The first lecture deals with curve fitting by means of Poisson's function. In the discrete case the approximating function is the product of a polynomial of degree n and of a Poisson probability; in the continuous case the Poisson factor is appropriately modified. The polynomial factor is written in terms of suitable orthogonal polynomials so that the method is convenient for computers. The topic of the second lecture is the Bernoulli distribution and the determination of its incomplete moments. E. Lukacs.

Prais, S. J., and Aitchison, J. The grouping of observations in regression analysis. Rev. Inst. Internat. Statist. 22 (1954), 1-22.

In cases in which a large number of observations are available it may be desirable to shorten the computation to carry out a linear regression analysis by grouping the observations. The authors carry out a mathematical examination of such a procedure, under the usual regression model with the assumption that each individual observation is replaced by the average of the group in which it is placed. Introducing grouping matrices, by the usual least squares method, it is shown that unbiased estimate of the regression coefficients and the residual variance are obtained. Some loss in efficiency necessarily results; it is shown that for the bivariate case this loss may be quite small in common situations. There is discussion of best ways to form groups; in general one should make them as homogeneous as possible as one would expect. The more general case of heteroscedasticity is treated more briefly under the assumption of zero covariances but that variances are proportional to the square of the expected value of the dependent variable. Unbiased estimates of the regression coefficients and the residual variance still result. There is discussion of ways of estimating an unknown variance matrix in this case and an iterative procedure is suggested. C. C. Craig.

Rao, C. Radhakrishna. Analysis of dispersion for multiply classified data with unequal numbers in cells. Sankhyā 15 (1955), 253-280.

The situation is considered where each of a number of individuals is classified in several ways, two or more measurements are made on it, and the resulting cells in the classification contain unequal numbers. It is desired to test the overall effects of the different classifications, their possible interactions and to estimate any effects found significant. The measurements are assumed to have expectations which are known linear functions of unknown parameters describing the classification; to have fixed, but unknown, dispersion matrices and independence between individuals. The method of solution is to regard the problem as one in least squares, evaluate the necessary residuals, test, and then solve the normal equations where required. The technique for the significance tests is the usual one of replacing the several variables by a linear function for which a variance ratio test is constructed:

the function being chosen to maximize this ratio. The major part of the paper describes and illustrates the computational schemes for two- and three-way classifications based on the above ideas. The appendix discusses the computational theory of least squares when the matrix of normal equations is singular. Both the Gauss-Doolittle and the superior square-root methods are mentioned and the singularity shown to present no difficulties. The reader interested in the computational details should beware of printing errors. *D. V. Lindley.*

Burr, Irving W. Calculation of exact sampling distribution of ranges from a discrete population. *Ann. Math. Statist.* 26 (1955), 530-532.

A method is developed for calculating the frequency distribution of ranges in samples of n values of a random variable whose range is the set of integers a to b , both finite. An example gives the distributions of ranges in samples of 5 from two populations, one of which approximates a normal distribution and the other a Pearson type III distribution with skewness 0.99. The third and fourth standard moments of the range distributions differ considerably less than in the universes sampled.

C. C. Craig (Ann Arbor, Mich.).

David, H. A. Moments of negative order and ratio-statistics. *J. Roy. Statist. Soc. Ser. B.* 17 (1955), 122-123.

Applications are made of the theorem, "If X and Y are independent random variables then $E[XY] = E[X]E[Y]$," to computing moments of statistics, e.g., the k th moment of Student's t can be found by letting $X^{1/2}$ and $Y^{-1/2}$ be the average and standard deviation of a sample from a normal population. *I. R. Savage (Stanford, Calif.).*

Dandekar, V. M. Certain modified forms of binomial and Poisson distributions. *Sankhyā* 15 (1955), 237-250.

The usual independence of binomial trials is modified essentially by imposing the condition that if a "success" has appeared at the i th "trial" it cannot occur again until at least the $(i+m)$ th "trial", $m > 0$. This type of process is encountered often. The author is essentially interested in obtaining the probability of x "successes" in n "trials" when: 1) the process starts at $i=0$, or, if $i > 0$, then the "past" is known; 2) one observes the process for the first time at some unknown stage. Neither case is handled in complete detail, the second case suffers from being presented in an enigmatic manner. The paper concludes by fitting modified Poisson distributions to several sets of data, the relevance of this material being questionable.

M. Muller (Ithaca, N.Y.).

Basu, D. A note on the structure of a stochastic model considered by V. M. Dandekar. *Sankhyā* 15 (1955), 251-252.

The note presents a more precise formulation and discussion of the modified binomial distribution than that given by Dandekar [see the preceding review]. The note also considers a familiar device which Dandekar did not utilize, namely, to use the probability generating function when studying a recurrence relation. *M. Muller.*

Rao, K. S. On the mutual independence of a set of Hotelling's T^2 derivable from a sample of size n from a k -variate normal population. *Bull. Inst. Internat. Statist.* 23 (1951), part II, 171-176.

"The purpose of the present paper is to show that from a sample of size n from a k -variate population with

means (m_1, \dots, m_k) and dispersion matrix (λ_{ij}) it is possible to derive $n-k-1$ 'Studentized' statistics each of which is independently distributed of the others as a Hotelling's T^2 . (Author's introduction.) *C. C. Craig.*

Haberman, Sol. Distributions of Kendall's tau based on partially ordered systems. *Biometrika* 42 (1955), 417-424.

The τ statistic has been introduced by M. G. Kendall in order to measure the consistency of the rankings of a set of n objects against some standard order or to measure the consistency of two rankings of n objects each. In Kendall's case, objects are ranked according to a single characteristic. In the present paper, n objects are ranked according to r characteristics, with each characteristic having p_i levels, $i=1, 2, \dots, r$, and with $n = \prod_{i=1}^r p_i$. When objects are ordered according to more than one characteristic, one generally has order relations among some, but not among all of the objects. Ordered subsets are called partial orderings. The problem is to measure the "closeness" of an observed ranking to one of the rankings which are consistent with the partial orderings. Closeness is measured by the minimum number of interchanges among adjacent pairs in order to attain consistency. The distribution of the number of interchanges for all possible rankings ($n!$) of n objects is given for the special cases: $r=2$, $p_1=2$, $p_2=2, 3, 4$; $r=2$, $p_1=p_2=3$ and $r=3$, $p_1=p_2=p_3=2$. *B. Epstein (Stanford, Calif.).*

Anderson, Oskar. Eine "nicht-parametrische" (verteilungsfreie) Ableitung der Streuung (variance) des multiplen (R_{xy}) und partiellen ($R_{xy \cdot z}$) Korrelationskoeffizienten im Falle der sogenannten Null-Hypothese, sowie der dieser Hypothese entsprechenden mittleren quadratischen Abweichungen (standard deviations) der Regressionskoeffizienten. *Mitteilungsbl. Math. Statist.* 7 (1955), 85-112.

Sampford, M. R. The truncated negative binomial distribution. *Biometrika* 42 (1955), 58-69.

If the random variable with the negative binomial probability distribution

$$P(r) = \{(k+r-1)! / [(k-1)! r!]\} p^r (1+p)^{-k-r},$$

for $r=0, 1, 2, \dots$; $p > 0$, $k > 0$, is truncated at 0, i.e. if values $r=0$ can not occur, then the truncated distribution can be written

$$P_1(r) = \{(k+r-1)! / [(k-1)! r!]\} \omega^k (1-\omega)^r / (1-\omega^k),$$

for $r=1, 2, \dots$, where $\omega = (1+p)^{-1}$. The problem of estimating the parameters ω , k is answered by a rather quickly converging procedure based on the method of moments and using trial and iteration as well as an auxiliary table given in the paper. The efficiency of this procedure is discussed. A procedure for solving the maximum-likelihood equations, alternative to that given by David and Johnson [*Biometrics* 8 (1952), 275-285; *MR* 14, 665] is also proposed. Numerical examples are presented to illustrate both procedures. *Z. W. Birnbaum.*

Weibull, Waloddi. New methods for computing parameters of complete or truncated distributions. *Flygtekn. Försöksanstalt. Rep.* 58 (1955), 21 pp.

Denote by $P(x)$ a distribution function; the author introduces the upper vertical i th moment

$$\bar{x}_i = \int_{x_i}^{\infty} (1-P)^i dx \text{ and } \bar{x}_i = \int_{x_i}^{\infty} P^i dx,$$

where x_0 and x_1 may have any values from $-\infty$ to ∞ . If $x_0 = -\infty$, and $x_1 = \infty$, then such vertical moments are called complete. Methods for the estimation of such moments from a sample and applications to various problems are given, particularly for the estimation of the parameters x_0 , x_1 , and a , $P=0$, for $x \leq x_0$, $P=1-e^{-W}$, where $W = \{(x-x_0)/x_0\}^{1/2}$, $x \geq x_0$. In the applications of lower vertical moments, it is claimed that $P=W$, $x \geq x_0$ is a distribution function which contradicts the definition of a distribution function for large x . The author does not make clear the advantages of such moments, either theoretically or from a numerical point of view.

L. A. Aroian (Culver City, Calif.).

Des Raj. On optimum selections from multivariate populations. *Sankhyā* 14 (1955), 363-366.

Let the random variable (Y, X_1, \dots, X_s) have the joint probability density $f(Y, X_1, \dots, X_s) = f_1(x)\Phi(Y|x)$ where x is an abbreviation for (X_1, \dots, X_s) , and let $\eta(x)$ be the regression function of Y on x . The author characterizes in terms of f , Φ , f_1 , and η the regions in x -space to which (Y, X_1, \dots, X_s) must be truncated in order that the marginal distribution of Y for the truncated population have certain desired properties, of which four types are considered. For three of these, if f is assumed multinormal the regions of truncation coincide with those obtained by Birnbaum and Chapman [*Ann. Math. Statist.* 21 (1950); 443-447; MR 12, 271] and by Cochran [*Proc. 2nd Berkeley Symposium Math. Statist. Probability*, 1950, Univ. of California Press, 1951, pp. 449-470; MR 13, 480]. The fourth type appears not to have been treated previously even for the multinormal case.

Z. W. Birnbaum (Seattle, Wash.).

Wurtele, Zivia S. A rectifying inspection plan. *J. Roy. Statist. Soc. Ser. B* 17 (1955), 124-127.

"In Suppl. *J. Roy. Statist. Soc.* 8 (1946), 216-222 [MR 9, 49], Anscombe discusses rectifying inspection plans which control the probability of using a substandard lot. A method is given for calculating the stopping points of one of these." (Author's summary.) J. Wolfowitz.

***da Silva Leme, Ruy Aguiar. Os extremos de amostras ocasionais e suas aplicações à engenharia. [The extremes of random samples and their applications in engineering.]** Thesis, University of São Paulo. São Paulo, 1954. x+167 pp.

The author summarizes the principle results concerning the probability distribution of the largest member x of a sample of n independent observations, and mentions their relevance to the theory of certain statistical tests of significance. He gives an account of the asymptotic behavior for large n , when the cumulative distribution function takes usually one of the forms $\exp(-u^{-k})$, $\exp(-v^k)$, $\exp(-e^u)$, where $u = \alpha(x-x_0)$ and $v = -u$. The major part of the monograph is devoted to a discussion of the relevance of these forms to various empirical laws, particularly these occurring in flood prediction and in the study of strength of materials. The author discusses alternatives to the theory of floods in the form in which it was left by Gumbel and concludes that Gumbel's theory is in fact well-adopted to practical requirements. After discussion of the studies of Weibull, of Daniels and of Tucker in the statistical theory of breakage, the author puts forward new mathematical models for plasto-ductile and plasto-brittle materials. He reformulates the concept of "safety factor" and proposes

that a new measure of safety should be adopted based not on the probability of breakdown but on the coefficient of variation of the strength of the material. The bibliography, which aims at being complete for years prior to 1954, gives 148 references.

A. R. G. Owen.

Cohen, A. Clifford, Jr. Censored samples from truncated normal distributions. *Biometrika* 42 (1955), 516-519.

The author finds maximum-likelihood estimates of the population mean and standard deviation based on censored samples drawn from truncated normal distributions. The asymptotic variances of the estimates are given. The results are useful in life testing and response time studies.

B. Epstein (Stanford, Calif.).

Huitson, A. A method of assigning confidence limits to linear combinations of variances. *Biometrika* 42 (1955), 471-479.

An approximate series expansion suitable for estimating confidence limits for a general linear combination of variances has been derived as far as terms of order $f_i^{-3/2}$, where f_i is the degrees of freedom for the i th variance. Tables of the upper and lower 5% and 1% critical points for the sum of two variances are given to two decimals for all combinations of (f_1, f_2) , where either f may assume the values of 16, 36, 144, ∞ .

L. A. Aroian.

Seal, Hilary L. The estimation of mortality and other decremental probabilities. *Skand. Aktuarietidskr.* 37 (1954), 137-162 (1955).

The author considers a specified group of individuals and assumes that two forces of decrement act on this group. He gives minimum variance unbiased estimates for the decremental probabilities corresponding to the two forces of decrement.

E. Lukacs (Washington D.C.).

Stuart, Alan. A paradox in statistical estimation. *Biometrika* 42 (1955), 527-529.

It is shown that the (asymptotic) efficiency of an estimator which is a function of two other correlated estimators may be decreased if one of the latter is replaced by its parameter value.

H. Teicher.

Lehmann, E. L., and Scheffé, Henry. Completeness, similar regions, and unbiased estimation. II. *Sankhyā* 15 (1955), 219-236.

This paper continues part I of the same title [*Sankhyā* 10 (1950), 305-340; MR 12, 511]. Part II deals with a family $\mathcal{M} = \{M_\theta \mid \theta \in \omega\}$ of measures, M_θ , which are absolutely continuous with respect to $(w.r.t.)$ a common σ -finite measure μ on an additive family of sets in the space W of points x . The label set ω is a Borel set in Euclidean space. The family \mathcal{M} is strongly complete if $\int_W f(x) dM_\theta = 0$ (a.e. w.r.t. Lebesgue measure) implies $f(x) = 0$ (a.e. \mathcal{M}). Strong completeness is a useful tool to get results about completeness (cf. part I). Techniques are given for determining strong completeness for families of product measures. Under regularity conditions, the strong completeness of the family of measures whose density functions w.r.t. μ are of the form

$$C(\psi, \theta_1, \dots, \theta_r) h(x) \exp [\psi(x) + \sum_{i=1}^r \theta_i t_i(x)]$$

is established. This, with the results of part I, is used to determine the existence of uniformly most powerful unbiased tests of various hypotheses about Ψ in the presence of the nuisance parameters $\theta_1, \dots, \theta_r$. There are a number of useful examples.

M. Dwass.

Epstein, Benjamin. Comparison of some non-parametric tests against normal alternatives with an application to life testing. *J. Amer. Statist. Assoc.* 50 (1955), 894-900.

Four non-parametric tests are compared experimentally regarding their power in detecting differences in means of two normal populations with common variance. The results indicate, for two samples of ten, that various exceedance and truncated maximum deviation criteria are superior to the run test but inferior to the rank sum test. However, the first two methods may be applied in truncated samples and their possible usefulness in life-tests is stressed.

H. A. David (Melbourne).

Page, E. S. A test for a change in a parameter occurring at an unknown point. *Biometrika* 42 (1955), 523-527.

Consider a sample of n independent observations. The order in which the observations were obtained is known and preserved. It is desired to test the null hypothesis that all n observations come from the same distribution $F(x|\theta)$ against the alternative that the first m observations ($m \leq n$) come from $F(x|\theta)$ while the remainder come from $F(x|\theta')$ ($\theta' \neq \theta$); m is unknown. The present paper considers a one-sided test for the special case that θ is a scale parameter. The numerical computations presented relating to the power of the proposed test are not too extensive. These data tend to refute the value of the test since it shows that the test is essentially not any more powerful than a purportedly (weak) classical test which assumes that all the observations are from the same distribution.

M. Muller (Ithaca, N.Y.).

Stuart, Alan. A test for homogeneity of the marginal distributions in a two-way classification. *Biometrika* 42 (1955), 412-416.

For two-way classifications in which the classes in the two margins are identical, often it may be desirable to test the two sets of observed marginal frequencies for homogeneity. For more than a 2×2 table an application of the likelihood principle runs into difficulties. The author proposes a test employing differences of corresponding marginal relative frequencies, which for large samples will tend to obey a multivariate normal distribution. On the null hypothesis the quadratic form in the exponent of such a distribution function is in the limit χ^2 with $n-1$ degrees of freedom for an $m \times m$ table. The elements of the matrix of this form are derived; these are estimated by equating observed values to their expected values and investing the resulting matrix. The χ^2 test obtained is illustrated with a numerical example.

C. C. Craig (Ann Arbor, Mich.).

Nandi, H. K. Joint tests of several hypotheses. *Calcutta Statist. Assoc. Bull.* 6 (1955), 17-31.

Let a random vector X have a distribution law given by the density function $f(X, \theta_1, \theta_2, \dots, \theta_r)$. Let us suppose that the k parameters are grouped into r mutually exclusive classes. A hypothesis H_i will involve the parameters of the i th class. The null hypothesis H^0 specifies $H_1^0, H_2^0, \dots, H_r^0$. Some theorems concerning the optimal nature of certain tests of H^0 are given. For example, one theorem deals essentially with the $(r+1)$ -action problem where either H^0 is accepted or rejected in favor of one of the H_i' , where H_i' states that the i th class of parameters differs from those specified under H^0 but the other groups coincide with those specified by the $H_j^0, j \neq i$. Here an optimal critical region of size α consists of the union

of optimal critical regions of certain sizes α_i for testing H^0 against H_i' . Another theorem deals with the 2nd-action problem, where either H^0 is accepted or some of the H_i^0 are rejected. This reviewer finds the statements and demonstrations in the paper somewhat confusing but considers that the type of problems stated and considered are of fundamental importance in applied work.

H. Chernoff (Stanford, Calif.).

Suryanarayana, O. Estimation and tests of significance of the components of a time-series. *Sankhyā* 15 (1955), 303-316.

The least-squares treatment of linear decomposition into trend, periodic component and random error.

H. Wold (Uppsala).

See also: Bose and Clatworthy, p. 227; Vartak, p. 227.

Mathematical Biology

Rashevsky, N. Some remarks on topological biology. *Bull. Math. Biophys.* 17 (1955), 207-218.

Rashevsky, N. Life, information theory, and topology. *Bull. Math. Biophys.* 17 (1955), 229-235.

Komatu, Yūsaku. Probability-theoretic investigations on inheritance. XVI. Further discussions on interchange of infants. *Proc. Japan Acad.* 28 (1952), 538-541, 542-545; 29 (1953), 36-41, 42-46.

[For parts III-XV see MR 17, 59-60.] Given two parents of known phenotype with respect to a system of allelomorphous genes A_1, A_2, \dots, A_m , and a putative child C_1 , and also a child C_2 whose putative mother is of known phenotype, an expression is derived (in terms of the population frequencies of the genes A_i on the assumption of random mating) for the probability that it will be possible from knowledge of their phenotypes to prove that the children C_1 and C_2 have been interchanged. This probability is also calculated for the case when C_1 has only one parent of known phenotype. Other variations of the problem are discussed.

On page 40 of the part published as *Proc. Japan Acad.* 29 (1953), 36-41, the author gives corrections to formulae which appeared incorrectly in the earlier part [ibid. 28 (1952), 538-541, 542-545]. The formulae are (2.6) to (2.17) on page 541 and (5.4) on page 545.

A. R. G. Owen (Cambridge, England).

Komatu, Yūsaku, and Nishimiya, Han. Lineal combinations on a Mendelian inherited character. *Rep. Statist. Appl. Res. Union Jap. Sci. Eng.* 3 (1953), 13-22.

Explicit formulae are derived for $\pi_{\alpha}(ij, kl)$, the joint probability that in an infinite population in equilibrium under random mating an individual will be of genotype $A_i A_j$, and an n th lineal descendant will be of genotype $A_k A_l$, where A_i ($i=1, 2, \dots, m$) is a system of multiple alleles.

A. R. G. Owen (Cambridge, England).

Komatu, Yūsaku. Further discussions on mother-child combinations concerning an inherited character after a panmixia. *Rep. Statist. Appl. Res. Union Jap. Sci. Eng.* 3 (1954), 42-53.

In an earlier paper [*J. Math. Soc. Japan* 6 (1954), 283-302; MR 16, 733] the author considered a population of N

males and N females, in equilibrium under random mating and segregating for the allelic series A_1, A_2, \dots, A_m . He obtained a generating function for the joint probability that a mother and her child should have respective genotypes $A_i A_j$ and $A_k A_l$, and derived the variance of the stochastic variable $X(ij, kl)$, the number of such combinations $A_i A_j, A_k A_l$. In the present paper he establishes the generating function by means of an argument involving a stochastic variable X_{ij} . He verifies his former results and in addition gives expressions for the mutual covariances of the $X(ij, kl)$.
A. R. G. Owen.

Komatu, Yûsaku. Alternative expressions for probability-generating functions concerning an inherited character after a panmixia. *Kôdai Math. Sem. Rep.* 1954, 43-54.

The author reviews the various mathematical methods available for calculating the probability generating function of the number of $A_i A_j$ genotypes, and of the parent-child combinations $A_i A_j, A_k A_l$, in a population of size $2N$ as considered by him in other papers [*J. Math. Soc. Japan* 6 (1954), 266-282, 283-302; *MR* 16, 733; see also the paper reviewed above].
A. R. G. Owen.

See also: A. Haimovici, p. 165; Waugh, p. 276.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Gluškov, V. M. Locally nilpotent locally bicomact groups. *Trudy Moskov. Mat. Obšč.* 4 (1955), 291-332. (Russian)

A group is locally nilpotent if every finitely generated subgroup is nilpotent. A connected locally compact locally nilpotent (LCLN) group is actually nilpotent, as follows rapidly from the fact that it is an inverse limit of Lie groups. This investigation is thus mainly of interest in the totally disconnected case, but most results are formulated for the general case. The techniques are a blend of abstract group theory (manipulation of commutators and central series) and point set topology (of the sort used in the structure theory of locally compact abelian groups). The following covers the highlights of the paper.

Lemma 1.8. In a non-discrete LCLN group the centralizer of any finite subset is non-discrete. Definition. An element is compact if the closure of the cyclic subgroup it generates is compact. Cor. 3.5.1. In any LN topological group the product of two compact elements is compact. Lemma 3.8. In any LCLN group the subgroup of compact elements is closed. Theorem 4.2. Let G be a connected LCLN group in which the only compact element is 1. Then G is a Lie group. Theorem 5.1. In any connected LCLN group the subgroup of compact elements is connected, compact, and central. The quotient is a Lie group. Theorem 8.3(a). If G is LCLN, K the component of the identity, and A the subgroup of compact elements, then KA is open and G/KA is torsion-free. In § 8.8.1 an example is given of a nilpotent LC group which is not an inverse limit of Lie groups. Lemma 9.2 and Theorem 9.7. In a LN group G generated by a compact neighborhood of 1, the subgroup of compact elements is compact. Any closed subgroup of G is again compactly generated. In § 10 an example is given of a centerless LN compact group. Theorem 11.3. If G is generated by a compact neighborhood of 1 and the transfinite ascending central series of G reaches G , then G is nilpotent. Theorem 11.4. If G is LN, locally connected, and generated by a compact neighborhood of 1, then G is nilpotent. Theorem 11.7. If G is LN and generated by a compact neighborhood of 1, then G is an inverse limit of Lie groups. The final section (§ 12) presents theorems of the Sylow type for totally disconnected LCLN groups.
I. Kaplansky (Chicago, Ill.).

Helgason, Sigurður. Some problems in the theory of almost periodic functions. *Math. Scand.* 3 (1955), 49-67.

Let G be a topological Abelian group in which, for every pair of distinct points x, y , there exists a continuous

character χ such that $\chi(x) \neq \chi(y)$. Let G^* denote the discrete group of continuous characters of G , and let \mathfrak{M} denote the Bohr mean value for continuous almost periodic functions on G . This paper is concerned with algebraic properties of the algebra A of continuous almost periodic functions on G , with pointwise addition and scalar multiplication, and with the product $x*y$ defined by $x*y(t) = \mathfrak{M}_s\{x(ts^{-1})y(s)\}$ for all $x, y \in A$. A major tool is naturally the Fourier transform \hat{x} , defined for all $x \in A$ as the function on G^* such that $\hat{x}(\chi) = \mathfrak{M}_t\{x(t)\chi(t)\}$. A function q on G^* is said to be a multiplier for A if $x \in A$ implies that $\hat{x}q = \hat{y}$ for some $y \in A$. Theorem: Multipliers for A are just those functions on G^* that are linear combinations of positive definite functions on G^* . Let Q be the endomorphism of A that corresponds to the multiplier q : for $x \in A$, Qx is the function y such that $\hat{Qx} = \hat{x}q$. Theorem: Q is an isometry of A if and only if q is a character of G^* multiplied by a number of absolute value 1. A distinguished set in G^* is a subset S of G^* such that, for all $x \in A$, $\hat{x}\varphi_S = \hat{y}$ for some $y \in A$, where φ_S is the characteristic function of the set S . Theorem: The distinguished sets in G^* carrying non-negative functions x into non-negative functions are just the subgroups of G^* . For other distinguished sets, the situation is more complicated.

Let σ be a one-to-one transformation of G^* onto itself. Suppose that for all $x \in A$, the functions on G^* whose values at χ are $\hat{x}(\sigma\chi)$ and $\hat{x}(\sigma^{-1}\chi)$ are Fourier transforms of elements of A . Then T_σ , where $(T_\sigma x)^\wedge(\chi) = \hat{x}(\sigma^{-1}\chi)$ for all $x \in A$, is clearly an automorphism of A . Theorem: If $\|T_\sigma\| = 1$, then $\sigma(\chi\psi)\sigma(1) = \sigma(\chi)\sigma(\psi)$ for all $\chi, \psi \in G^*$, and T_σ is an isometric automorphism of A . [An analogous theorem for L_1 of a locally compact Abelian group was proved by Helson, *Ark. Mat.* 2 (1952), 475-487; *MR* 15, 327.] In passing, the author re-obtains a formula of Eberlein [*Proc. Amer. Math. Soc.* 6 (1955), 310-312; *MR* 16, 817] and a remark of the reviewer [*Trans. Amer. Math. Soc.* 74 (1953), 303-322; *MR* 14, 882]. Line 16 of page 52 seems obscure, although the inequality is correct. On page 57, line 9, read "1" for "i".
E. Hewitt.

Eberlein, W. F. Characterizations of Fourier-Stieltjes transforms. *Duke Math. J.* 22 (1955), 465-468.

The paper extends to functions on any locally compact Abelian group G a characterization of Fourier-Stieltjes transforms which was given for the real line by Bochner [*Bull. Amer. Math. Soc.* 40 (1934), 271-276]. The theorem proved is that if φ is a measurable function on G , and if there is a constant M such that

$$|\sum a_n \varphi(x_n)| \leq M \sup_{\sigma \in G} |\sum a_n(x_n, \sigma)|$$

for all finite sets of complex numbers (a_n) and points (x_n) in G , then there is a unique Radon measure $\tilde{\mu}$ on \tilde{G} such that $\varphi(x) = \int \tilde{\varphi}(x, \tilde{x}) d\tilde{\mu}$ almost everywhere, and the total variation of φ is the least M for which the above inequality holds.

J. L. B. Cooper (Cardiff).

Faucett, W. M., Koch, R. J., and Numakura, K. Completions of maximal ideals in compact semigroups. Duke Math. J. 22 (1955), 655-661.

Let S be a semigroup (i.e., a set with an associative multiplication). A set TCS is an ideal if $ST \cup TS \subset T$. The Rees quotient S/T is the set $S \cap T' \cup \{T\}$, with $x\{T\} = \{T\}$, $\{T\}^2 = \{T\}$, $xy = \{T\}$ if $xy \in T$, and xy as in S if $xy \notin T$. Suppose from now on that S is a compact mob (a mob is a Hausdorff space that is a semigroup in which multiplication is continuous). Theorem 1. Let J be a maximal proper ideal in S . Then S/J is the trivial semigroup $\{0, a\}$ with all products 0, or S/J has no proper ideals and contains an idempotent element that is not a two-sided unit for any other idempotent. Theorem 2. $S \cap J'$ is the union of pairwise disjoint compact groups and of compact sets A_α such that $A_\alpha^2 \subset J$ (one of these types may be absent). Theorem 3. Suppose that $S^2 = S$, and let $\tilde{E} = \{x: x \in S, x^2 = x, x \in SxS, \text{ and } y \in SxS \text{ whenever } x \in SyS\}$. Then $S = S\tilde{E}S$. Several other theorems of this general kind are also proved.

E. Hewitt (Princeton, N.J.).

See also: Shields, p. 245; Kadison, p. 285; Wechsler, p. 287.

Lie Groups, Lie Algebras

Borel, Armand. Topology of Lie groups and characteristic classes. Bull. Amer. Math. Soc. 61 (1955), 397-432.

The present paper surveys recent advances in the study of the topology of Lie groups and constitutes a sequel to Samelson's report [same Bull. 58 (1952), 2-37; MR 13, 533]. We outline the subjects reported: Generalizations of Hopf's and Samelson's theorems (to the effect that the rational cohomology ring and Pontrjagin ring of a compact connected Lie group are exterior algebras) to coefficient fields of characteristic > 0 . Review of fundamental facts on characteristic classes (Stiefel-Whitney, Pontrjagin, Chern). Cohomology of classifying spaces for coefficient fields of arbitrary characteristic, connection between these cohomology groups and the invariants of the Weyl group. Results on the cohomology of compact simple groups for fields of various characteristics, results on the Steenrod squares in Lie groups, connections between the cohomology of Lie groups and the Weyl group and other group-theoretic properties. Cohomology and characteristic classes of homogeneous spaces (various results on the characteristic classes of G/T hinted at here have not been published yet). Homogeneous complex spaces, homogeneous spaces of non-compact Lie groups. Review of the results on the homotopy of Lie groups obtained hitherto. An extensive bibliography concludes this excellent survey. W. T. van Est (Amsterdam).

Harish-Chandra. Representations of semisimple Lie groups. IV. Amer. J. Math. 77 (1955), 743-777.

This is a continuation of the author's series of papers on representations of semi-simple Lie groups. [Trans. Amer.

Math. Soc. 75 (1953), 185-243; 76 (1954), 26-65, 234-253; MR 15, 100, 398; 16, 11]. Let G be a non-compact simple Lie group whose first Betti number is one. Assuming that G has finite center and K is a maximal compact subgroup, then G/K is naturally isomorphic to a bounded symmetric domain in some complex affine space [see E. Cartan, Abh. Math. Sem. Hamburg. Univ. 11 (1935), 116-162]. This fact is closely related with the series of unitary representation of G which the author studies in the present paper. They are naturally parametrized by the characters of a compact Cartan subgroup of G . Thus these representations generalize to higher dimensions the discrete series of unitary representations of $SL(2, R)$ due to Bargmann [Ann. of Math. (2) 48 (1947), 568-640; MR 9, 133]. The present paper is mainly concerned with algebraic properties of these representations of G , such as corresponding properties of the roots and weights; their function-theoretic properties will appear elsewhere. Some of the results of the present paper have been outlined in Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 314-317 [MR 17, 60]. F. I. Mautner (Princeton, N.J.).

Hochschild, G. Restricted Lie algebras and simple associative algebras of characteristic p . Trans. Amer. Math. Soc. 80 (1955), 135-147.

Dans un travail récent [mêmes Trans. 79 (1955), 477-489; MR 17, 61], l'auteur a montré comment on peut caractériser au moyen des algèbres de Lie „restreintes” les algèbres simples de centre F admettant un sous-corps commutatif maximal C qui soit une extension purement inséparable de F d'exposant 1. Il étend ici ses méthodes au cas plus général suivant: C étant une extension purement inséparable d'exposant 1 de F , F le centre d'une algèbre simple B , soit A l'algèbre commutante de C dans B , qui est simple et a pour centre C . Soit S l'ensemble des $s \in B$ tels que $D_s(c) = sc - cs \in C$ pour tout $c \in C$, et soit $\varphi(s)$ la restriction de D_s à C . S est une algèbre de Lie restreinte sur C , φ un homomorphisme de S sur l'algèbre T des dérivations de C s'annulant dans F , et le noyau de φ est A ; on montre en outre que l'extension (S, φ) de A est „régulière” au sens défini par l'auteur (voir le compte rendu du travail précité). Le théorème principal de l'auteur constitue encore une réciproque des résultats précédents, en associant à toute extension régulière (S, φ) d'une algèbre simple A de centre C une algèbre simple B de centre F dans laquelle A et C sont commutantes l'une de l'autre; il y parvient en étendant convenablement les constructions déjà utilisées dans le cas particulier $A=C$. Il s'agit ensuite, étant données deux extensions régulières (S, φ) , (S', φ') , correspondant aux algèbres simples B, B' de centre F , de décrire l'extension régulière (E, ψ) qui correspond au produit tensoriel $B \otimes_F B'$; c'est ce que l'auteur appelle la composée de (S, φ) et (S', φ') , et dont il donne explicitement une définition assez compliquée. S'appuyant d'abord sur des résultats classiques d'Albert-Hasse, l'auteur remarque ensuite (th. 5) que si C est une extension purement inséparable de F d'exposant quelconque, toute algèbre simple A de centre C peut s'écrire $B \otimes_F C$, où B est une algèbre simple de centre F ; il montre d'autre part (th. 6) que toute dérivation d'une sous-algèbre semi-simple $A' \subset C$ de A , transformant C en lui-même, s'étend à une dérivation de A (généralisation d'un théorème de Jacobson); enfin, utilisant la théorie des extensions régulières, il remarque que le th. 5 peut aussi se démontrer en partant du th. 6, sans se servir de la théorie de Albert-Hasse. J. Dieudonné.

Chevalley, Claude, and Tuan, Hsio-Fu. Algebraic Lie algebras and their invariants. J. Chinese Math. Soc. (N.S.) 1 (1951), 215-242. (Chinese summary)

This is the original account of the theory of algebraic Lie algebras, as obtained by the authors in 1945 [Proc. Nat. Acad. Sci. U.S.A. 31 (1945), 195-196; MR 7, 4]. The main result is that a Lie algebra of linear transformations over a field of characteristic 0 is the Lie algebra of an algebraic group if and only if, with every linear transformation, it also contains all its replicas. (Actually, the statement refers to the field of the complex numbers; Lie algebras of linear algebraic groups over arbitrary fields were defined only after the writing of the present paper.)

Much of the content of this paper has since been absorbed in Chevalley's *Théorie des groupes de Lie*, vol. II [Hermann, Paris, 1951, MR 14, 448] which, in particular, contains the result stated above implicitly. Other proofs of the main result have been published by M. Gotô [J. Math. Soc. Japan 1 (1948), 29-45; MR 10, 426] and Y. Matsushima [ibid. 1 (1948), 46-57; MR 10, 426].

G. P. Hochschild (Berkeley, Calif.).

Schafer, R. D., and Tomber, M. L. On a simple Lie algebra of characteristic 2. Mem. Amer. Math. Soc. no. 14 (1955), 11-14.

Let \mathfrak{E} be a Cayley algebra over a field F , and \mathfrak{J} the set of hermitian 3×3 matrices with coefficients in \mathfrak{E} . Then \mathfrak{J} is a Jordan algebra under the law of composition $(X, Y) \rightarrow XY + YX$; if F is of characteristic $\neq 2$, this Jordan algebra is not special. If, on the contrary, F is of characteristic 2 (as will be assumed from now on), it is shown that \mathfrak{J} is a special Jordan algebra and is a restricted Lie algebra (the square operation of the restricted structure of \mathfrak{J} being $X \rightarrow X^2$). The derived algebra \mathfrak{J}_0 of \mathfrak{J} is simple; if F is algebraically closed, it is isomorphic to a previously known simple restricted Lie algebra. It is proved that the algebra of outer derivations of \mathfrak{J}_0 is simple of the same dimension 26 as \mathfrak{J}_0 itself; if F is algebraically closed, the algebra of outer derivations is actually isomorphic to \mathfrak{J}_0 .

C. Chevalley (Paris).

Tôgô, Shigeaki. On splittable linear Lie algebras. J. Sci. Hiroshima Univ. Ser. A. 18 (1955), 289-306.

Malcev's concept of splittability for Lie algebras of linear transformations (such an algebra being called splittable if, with every linear transformation, it also contains its semisimple and nilpotent components) is explored quite thoroughly, in parallelism with the known theory of algebraic Lie algebras. For example, a structural characterization (referring to the radical) of splittable Lie algebras is given which closely resembles Chevalley's structure theorem for algebraic Lie algebras [Ann. of Math. (2) 48 (1947), 91-100, p. 98; MR 8, 435]. It is shown also that an abstract Lie algebra over a field of characteristic 0 has a faithful splittable representation if and only if its adjoint representation is splittable.

The corresponding notion for groups of linear transformations is introduced, and it is shown that, if the base field is the field of the real or complex numbers, a group of linear transformations is splittable if and only if its Lie algebra is splittable.

G. Hochschild.

See also: Schafer, p. 232.

Topological Vector Spaces

Shibata, Toshio. On vector valued functions. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 20-32.

This paper notes the fact that, for functions of a complex variable with values in a complete, complex, locally-convex topological linear space (with Hausdorff separation) many theorems of classical analysis, especially those about series and integrals of continuous functions and those about analytic functions, can be proved by adaptation of classical methods or by use of linear functionals. The program here described is for the most part regarded as known by specialists in functional analysis. If the author had made use of semi-norms he could have abbreviated his discussions by remarking that many of the proofs are essentially the same as in the Banach space case. It may be noted that the theory of analytic functions with values in locally-convex topological linear spaces has been considered explicitly by A. Grothendieck [J. Reine Angew. Math. 192 (1953), 35-64; MR 15, 438]. The culmination of the paper is the theorem that if X is a complex topological algebra in which inversion is continuous, it is isomorphic to the complex number field. The proof is as in the case of Banach algebras, i.e., via Liouville's theorem. This result has perhaps never been explicitly published before, but knowledge of it seems to have considerable currency. A. E. Taylor.

Edwards, R. E. On factor functions. Pacific J. Math. 5 (1955), 367-378.

For a locally compact abelian group G , E and F are topological vector spaces of functions, measures, or distributions on G , with associated notions of Fourier transforms, denoted with "...". A function φ on the dual group G is a factor function in case $f \in E \Rightarrow \varphi \cdot f = g$, some $g \in F$. E and F are assumed to be translation-invariant. (E, F) denotes the class of factor functions; $L_1(E, F)$ denotes the set of continuous linear mappings which commute with translation. For $\varphi \in (E, F)$, define $u \in L_1(E, F)$ by $g = u(f)$ if $g = \varphi \cdot f$. Thus the more general class $L_1(E, F)$ is chosen for investigation. Results: 1. For $1 \leq p \leq \infty$, $u \in L_1(L^p, L^p) \Leftrightarrow u(f) = \mu * f$, where μ is a bounded Radon measure if $p=1$ and $\mu \in L^p$ if $1 < p \leq \infty$. 2. For the L. Schwartz space \mathcal{S} , $u \in L_1(\mathcal{S}, \mathcal{S}) \Leftrightarrow u(f) = \mu * f$, μ a rapidly decreasing distribution. Extensions of these results are made to a large collection of pairs of spaces. Proofs are indicated and sketched in many places without complete exposition.

B. Gelbaum (Minneapolis, Minn.).

Allen, H. S. Linear transformations and infinite matrices. J. London Math. Soc. 30 (1955), 501-504.

Let σ be the vector space of all complex sequences, and ϕ be the subspace of those that are zero off finite sets. Let α be a subspace of σ and let $(\alpha)^*$ denote the (algebraic) dual space of all linear functionals on α . The functionals L which have the form $L(x) = \sum y_n x_n$ (absolutely convergent) form a subspace $\alpha^* \subseteq (\alpha)^*$ called the Köthe-Toeplitz dual space for α . Let $\mathcal{L}(\alpha)$ be the algebra of all linear transformations of α into itself, and let $\Sigma(\alpha)$ be the subspace consisting of the $T \in \mathcal{L}(\alpha)$ which have the form $T(x) = y$ where $y_n = \sum a_{nk} x_k$ (absolutely convergent) for some infinite matrix $A = (a_{nk})$. Theorems 1&4: $(\phi)^* = \phi^*$ and $\mathcal{L}(\phi) = \Sigma(\phi)$; however, if ϕ is a proper subspace of α , α^* and $\Sigma(\alpha)$ are proper subspaces of $(\alpha)^*$ and $\mathcal{L}(\alpha)$. Let β be any subspace of α^* , and let $\mathcal{L}(\alpha|\beta)$ be the set of $T \in \mathcal{L}(\alpha)$ which have an adjoint, in the natural pairing of α and β . Likewise, let $\Sigma(\alpha|\beta)$ be the set of $T \in \Sigma(\alpha)$ whose transpose

(adjoint) lies in $\Sigma(\beta)$. Theorem 2: If α and β both contain ϕ , then $\mathcal{L}(\alpha|\beta) = \Sigma(\alpha|\beta)$. Theorem 3: if α contains ϕ , then $\Sigma(\alpha) = \mathcal{L}(\alpha|\alpha^*)$. [Reviewer's remarks: the author's results follow from the following observations: (i) any $L \in \alpha^*$ is completely determined by its values on ϕ ; (ii) any $T \in \Sigma(\alpha)$ is completely determined by its values on ϕ ; (iii) ϕ is dense in α in the weak topology defined by β ; (iv) if $T \in \mathcal{L}(\alpha|\beta)$, T is continuous in the β topology on α , and T' is continuous in the α topology on β .] *R. C. Buck.*

Allen, H. S., and Green, H. F. Existence theorems for reciprocals of infinite matrices belonging to rings of transformations. *J. London Math. Soc.* **30** (1955), 504-507.

The notation is the same as that of the preceding review. Let $\phi \in \alpha$ and let A be a matrix corresponding to a transformation T in the class $\Sigma(\alpha|\beta)$. Then, A has an inverse A^{-1} in the same class if and only if T maps α onto α , and its adjoint T' maps β onto β . Several additional theorems are derived from this, and a number of special cases are given. [Remark: The reviewer believes that a simpler proof may be obtained by using the following general theorem: let E and F be paired linear spaces and let $\mathcal{L}(E|F)$ be the set of transformations $T \in \mathcal{L}(E)$ having an adjoint T' in $\mathcal{L}(F)$. Then, T has an inverse in $\mathcal{L}(E|F)$ if and only if T and T' are both onto. Proof: T and T' are both one-to-one so that both have inverses. Moreover, the adjoint of T^{-1} is $(T')^{-1}$ so that $T^{-1} \in \mathcal{L}(E|F)$.] *R. C. Buck (Madison Wis.).*

Banach Spaces, Banach Algebras

Gol'dman, M. A. On the stability of the property of normal solvability of linear equations. *Dokl. Akad. Nauk SSSR (N.S.)* **100** (1955), 201-204. (Russian)

The author establishes precise conditions in order that a bounded linear operator T on one Banach space E_1 whose range TE_1 in another Banach space E_2 is closed, should be stable in regard to the latter property, i.e. should belong to the open kernel Γ_0 of the set Γ of such operators. Denoting as usual by $\alpha(T)$, $\beta(T)$ the dimensionalities of the null-spaces of T , T^* , it is shown to be necessary and sufficient that at least one of $\alpha(T)$, $\beta(T)$ should be finite. The necessity is new, and is based on a result of G. W. Mackey [*Bull. Amer. Math. Soc.* **52** (1946), 322-325; *MR* **7**, 455] on quasi-complements. The sufficiency extends results of the reviewer [*Mat. Sb. N.S.* **28(70)** (1951), 3-14; *Acta Sci. Math. Szeged* **15** (1953), 38-56; *MR* **13**, 46; **15**, 134] and is based on representations of T in the forms $U+K$, $U+V$ where either U or U^* is reversible and $U \in \Gamma$, K is finite-dimensional and V is completely continuous. If $T \in \Gamma$ and either $\alpha(T) < \infty$ or $\beta(T) < \infty$ or both, then either form of representation is possible; the converse statement is also proved. *F. V. Atkinson.*

Gohberg, I. C. On zeros and zero elements of unbounded operators. *Dokl. Akad. Nauk SSSR (N.S.)* **101** (1955), 9-12. (Russian)

The author announces, without proofs, three groups of results for a closed (not necessarily bounded) linear operator A on a Banach space E into itself, assuming also that A is normally soluble and that $\alpha(A) < \infty$ [see the preceding review]. First considered are necessary and sufficient conditions for A to have only a finite number of linearly independent zero-elements, that is to say $x \in E$

such that $A^*x=0$ for some n . One such condition is that $\alpha(A) \leq \beta(A)$ for every "part" A of A , acting on a sub-space \mathcal{E} of E into \mathcal{E} . If also $\beta(A) < \infty$, another such condition is $A=D+T$, where T is completely continuous and D has a bounded left inverse such that $DTD^{-1}=T$; this extends a result of M. A. Gol'dman and S. N. Kračkovskii [same *Dokl. (N.S.)* **86** (1952), 15-17; *MR* **14**, 478] for bounded A . Next considered are perturbations of A . If T is completely continuous, then $A+T$ is normally soluble and $\alpha(A+T) < \infty$. If B is small enough then $A+B$ is normally soluble and $\alpha(A+B) \leq \alpha(A)$. [For related results see the preceding review and references there given, and also B. Sz.-Nagy, *Acta Math. Acad. Sci. Hungar.* **3** (1952), 49-52; *MR* **14**, 564; and the author, *Mat. Sb. N.S.* **33(75)** (1953), 193-198; *MR* **15**, 233.] Finally, supposing E to be a Hilbert space, he extends results of S. N. Kračkovskii [*Dokl. Akad. Nauk SSSR (N.S.)* **88** (1953), 201-204; **91** (1953), 1011-1013; *MR* **14**, 1095; **15**, 437] concerning the canonical arrangement of zero-elements of $A-\lambda I$ into finite and infinite rows; Kračkovskii had assumed A to be bounded and $\beta(A)$ (as well as $\alpha(A)$) to be finite. *F. V. Atkinson (Canberra).*

Graves, Lawrence M. A generalization of the Riesz theory of completely continuous transformations.

Trans. Amer. Math. Soc. **79** (1955), 141-149.

The author extends the theory of F. Riesz [*Acta Math.* **41** (1916), 71-98] and some of its later developments to the case where the domain and the range need not lie in the same space. Let E, K be linear transformations of one Banach space \mathfrak{X} into another, \mathfrak{Y} , where $E\mathfrak{X}=\mathfrak{Y}$ but E is not one-to-one (i.e., in the notation of the two previously reviewed papers, $\alpha(E)>0$, $\beta(E)=0$), and K is completely continuous; he then studies the properties of $T_c=E-cK$, where c is a scalar. The definitions of null-spaces and invariant sub-spaces are extended to this case; proper values of T_c are such that $T_c\mathfrak{X} \neq \mathfrak{Y}$ (i.e. such that $\beta(T_c)>0$). The first four results extend the original Riesz theory, culminating in the property that the proper values have no finite limit point. The next result is that T_c^{-1} depends continuously on c except at proper values. The behaviour of T_c^* near a proper value is used to define the order of the proper value; this is shown to agree with a property of the invariant sub-spaces. Finally the author establishes a decomposition $K=G+H$ relative to a proper value; in particular, G has only this proper value, while H has all remaining proper values in common with K . [Reviewer's remark: There is slight contact with the paper of Gol'dman reviewed second above, in particular the facts that $T_c\mathfrak{X}$ is closed and that $\beta(T_c) < \infty$; the results and methods of the two papers are, however, mostly distinct.] *F. V. Atkinson (Canberra).*

Audin, Maurice. Sur le développement de certaines transformations linéaires en série de transformations orthogonales et de rangs finis dans un espace de Banach. *C. R. Acad. Sci. Paris* **240** (1955), 832-835.

In proving expansions associated with the discrete spectrum of a bounded linear operator A on a Banach space \mathcal{E} into itself, the difficulty arises of proving the convergence of certain monotonic sequences of projections. The author discusses the application to this point of theorems such as those of W. Orlicz [*S. Banach, Théorie des opérations linéaires, Warsaw, 1932*] wherein the boundedness of the sequence is postulated and \mathcal{E} is to be weakly complete. Other conditions are also given, but without proofs. [Reviewer's note: The reference

appears to be to p. 240 of Banach's book; p. 237 is also relevant.]
F. V. Atkinson (Canberra).

★Luxemburg, Wilhelmus Anthonius Josephus. **Banach function spaces.** Thesis, Technische Hogeschool te Delft, 1955. 70 pp.

The first part of this thesis examines a "general-special" class of Banach spaces, a typical member of which is a linear subclass X of the complex-valued measurable functions on a measure space Δ . A norm $\| \cdot \|_X$ is defined and related to the measure so that $f \in X$ if and only if $\|f\|_X < \infty$. The defining properties of the norm are designed to reflect in abstract form familiar facts of L_p . A particular subset X' (the associate space of X) of X^* (the conjugate space of X) is singled out by $g \in X'$ if and only if $\sup \{ \int |fg| d\mu \mid \|f\|_X \leq 1 \} < \infty$. This is equivalent, it seems to the reviewer, to singling out the absolutely continuous measures from among the set of all measures (linear functionals). One result is universal "reflexivity", i.e., $X = (X')'$.

$\|f\|_X$ is called absolutely continuous if, modulo the usual measure-theoretic hypotheses, $\mu(E_n) \rightarrow 0$ implies $\|f\|_{X_{E_n}} \rightarrow 0$. X^∞ , the set of elements of X having absolutely continuous norms, is a closed subspace; X^* , the set of elements $f \in X$ for which $\{x \mid f(x) \neq 0\}$ is contained in one of a fixed monotone sequence Δ_n of sets of positive finite measure for which $\bigcup \Delta_n = \Delta$, is also a closed subspace. Results: (1) $X^\infty = X^*$ if and only if $(X^*)^*$ and X' are isometric-isomorphic. (2) $X^* = X'$ isometrically if and only if $X = X^\infty = X^*$. (3) $X = X^{**}$ if and only if X and X' have absolutely continuous norms. (4) (In the Eberlein-Smulian line) X^* separable implies X is reflexive.

In the second part of the thesis, the discussion is specialized and the results of the first parts applied to Orlicz spaces. B. R. Gelbaum (Minneapolis, Minn.).

Tandori, Károly. **Über einen speziellen Banachschen Raum.** Publ. Math. Debrecen 3 (1954), 263-268 (1955).

Let $p \geq 1$ be fixed and consider the class $L_{0,p}$ of the functions $f \in L^p(0, 1)$ for which $x=0$ is a Lebesgue point of order p , that is

$$\int_0^h |f(t)|^p dt = o(h) \quad (h \rightarrow +0).$$

If we introduce the norm

$$\|f\|_p^p = \sup_{0 < h \leq 1} \left(h^{-1} \int_0^h |f(t)|^p dt \right)^{1/p},$$

the class becomes a Banach space, first considered by Korenblyum, Kreĭn and Levin [Dokl. Akad. Nauk SSSR (N.S.) 62 (1948), 17-20; MR 10, 306]. They also determined (loc. cit.) the general form of the linear functional in the space when $p > 1$. The author extends the result to $p=1$ and obtains the following theorem: The general linear operation $\Phi(f)$ in $L_{0,1}$ is of the form $\Phi(f) = \int_0^1 f(t) \varphi(t) dt$, where $\varphi(t)$ is measurable, essentially bounded in each interval $(s, 1)$, $s > 0$, and if $\psi(s)$ denotes the essential upper bound of $|\varphi|$ in $(s, 1)$, then the integral $\int_0^1 \psi(s) ds$ is finite. Moreover, the norm of the functional Φ is given by the last integral. A. Zygmund (Chicago, Ill.).

Kadison, Richard V. **On the orthogonalization of operator representations.** Amer. J. Math. 77 (1955), 600-620.

The problem considered here is the one raised by the following question: "Is a given representation (not neces-

sarily adjoint-preserving) of a C^* -algebra as bounded operators on a Hilbert space similar to a $*$ -representation?" A closely related question for group representations is the following: "Is a given bounded representation of a group by bounded operators on a Hilbert space similar to a unitary representation?" The author obtains necessary and sufficient conditions on representations for an affirmative answer to these questions. For example, a representation $g \rightarrow A_g$ of a group G by bounded operators on a Hilbert space H is similar to a unitary representation if and only if it is "boundedly locally semi-simple" in the following sense: For each finite set $x_1, \dots, x_n \in H$ and $g_1, \dots, g_n \in G$ there exists a constant M and a linear transformation S defined in the finite-dimensional space \mathcal{U} generated by $x_1, \dots, x_n, A_{g_1}x_1, \dots, A_{g_n}x_n$ such that $\|Sx_i\| = \|SA_{g_i}x_i\|$ ($i=1, \dots, n$) and $M^{-1} \leq \|Sx\| \leq M$ for all $x \in \mathcal{U}$ with $\|x\|=1$. A representation φ of a C^* -algebra \mathfrak{A} is similar to a $*$ -representation of \mathfrak{A} if and only if the restriction of φ to the group U of unitary operators in \mathfrak{A} is a boundedly locally semi-simple representation of U . The author also gives a definition (too complicated to be included here) of a "self-adjoint cover" of a representation φ of a C^* -algebra and shows that its existence is necessary and sufficient for φ to be similar to a $*$ -representation. Ehrenpreis and Mautner [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 231-233; MR 17, 126] have recently given an example of a group with a bounded representation which is not similar to a unitary representation. C. E. Rickart.

Kadison, Richard V. **Multiplicity theory for operator algebras.** Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 169-173.

A C^* -algebra is a self-adjoint, uniformly closed algebra of operators on a Hilbert space. In an earlier paper [Ann. of Math. (2) 56 (1952), 494-503; MR 14, 481] the author obtained "algebraic invariants" which determine when two C^* -algebras are algebraically isomorphic. In the present paper the author obtains "spatial (or multiplicity or unitary) invariants" for general C^* -algebras which also take into account the Hilbert spaces on which they act. The algebras are all assumed to possess an identity element. A positive functional on a C^* -algebra \mathfrak{A} which has the value 1 on the identity is called a "state". The states form a compact, convex subset of the space of all linear continuous functionals and the extreme points of this set are called "pure states". The closure of the pure states is a compact Hausdorff space called the "pure state space". If \mathfrak{A} is any C^* -algebra and φ is a representation of \mathfrak{A} as an algebra of operators on a Hilbert space \mathfrak{H} , then a "multiplicity function" f_φ is defined having all positive or zero real numbers plus the infinite cardinals as its domain and a certain descending class of ideals of Borel sets in the pure state space as its range. This multiplicity function is the desired system of spatial invariants as shown by the following theorem: Two representations φ_1, φ_2 of a C^* -algebra \mathfrak{A} as algebras $\mathfrak{A}_1, \mathfrak{A}_2$ acting on Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ are unitarily equivalent (i.e. there exists a unitary transformation U of \mathfrak{H}_1 onto \mathfrak{H}_2 such that $U\varphi_1(A)U^{-1} = \varphi_2(A)$ for all $A \in \mathfrak{A}$) if and only if their associated multiplicity functions are identical. This result reduces directly to the known result for commutative C^* -algebras. No proofs are given here since a detailed account will appear elsewhere. C. E. Rickart.

See also: Nakamura, p. 246; Žautykov, p. 270.

Hilbert Space

Slobodyanskii, M. G. On estimates for eigenvalues of a self-adjoint operator. Prikl. Mat. Meh. 19 (1955), 295-314. (Russian)

With a view to applications to differential equations, the author wishes to estimate from above and below the eigen-values of an unbounded self-adjoint linear operator A on a Hilbert space, whose inverse is completely continuous. The method involves the construction of two other operators, A_1 and A_2 , which also have completely continuous inverses. The eigen-values of A_1 provide the approximations to those of A , while A_2 seems to act as a bridge between A and A_1 . First given are the limits for the discrepancies between the two sets of eigen-values (allowing also for multiple eigen-values); this is the only formal result, the bulk of the paper being devoted to a discussion of the choice of A_1 and A_2 . A_1^{-1} is taken to be finite-dimensional and is specified in terms of a chosen set of elements of the domain of A . The choice of A_2 appears more difficult, but is completed in the case of a self-adjoint differential operator of order $2s$, with null boundary conditions. A worked example is $-u'' + (x-1.5)u - \lambda u = 0$, $u(0) = u(1) = 0$; somewhat lengthy calculations fix the first eigen-value to within about 5%. A simplified method, cutting out A_2 , is then given for the case when A is positive-definite, and is applied to the 2st order self-adjoint problem, and also to the vibrations of a clamped plate. Finally, the problem of the eigen-values of an operator of the form $A - \lambda B$ is briefly referred

to. A bibliography gives some useful early references. F. V. Atkinson (Canberra).

Temple, G. An elementary proof of Kato's lemma. Mathematika 2 (1955), 39-41.

T. Kato's estimate of eigenvalues [cf. J. Phys. Soc. Japan 4 (1949), 334-339; MR 12, 447] rests on the following lemma: If H is selfadjoint and has no spectrum in the closed interval $[\alpha, \beta]$, then $(H - \alpha I)(H - \beta I)$ is positive definite. The author exhibits an elementary proof of it, that does not rest on the spectral decomposition theorem, but on Bateman's "energy function" $f(\lambda)$ [cf. Trans. Cambridge Philos. Soc. 20 (1908), 371-382] defined by $((H - \lambda)^{-1}u, u)$. He shows in a simple way that it is real, with positive derivative and deduces from this the above lemma. František Wolf (Rome).

Bahadur, R. R. Measurable subspaces and subalgebras. Proc. Amer. Math. Soc. 6 (1955), 565-570.

Let (X, S, μ) be a probability space and $V = L_2(X, S, \mu)$ be the real Hilbert space of square integrable functions on X . It is proved that a necessary and sufficient condition for an orthogonal projection T on V to have range $L_2(X, S^*, \mu)$ for some σ -subalgebra S^* of S is T being constant-preserving and $Tf \geq 0$ whenever $f \geq 0$. As a corollary the same condition is a necessary and sufficient condition for T to be a conditional expectation operator. An application to finite-dimensional symmetric probability matrices is also indicated. S. C. Moy.

See also: Kadison, p. 285.

TOPOLOGY

Aleksandrov, P. S. On the concept of space in topology. Acta Math. Acad. Sci. Hungar. 5 (1954), supplementum, 43-60. (Russian)

An expository article. Some topics considered: metrizable spaces, dyadic compact spaces, limits of directed systems of spaces, proximity structures. M. Katětov (Prague).

Kac, G. I. Topological spaces in which one may introduce a complete uniform structure. Dokl. Akad. Nauk SSSR (N.S.) 99 (1954), 897-900. (Russian)

Let G be a completely regular (c.r.) topological space; if there exists a complete uniformity structure on G , then G is called a "P-space". The principal theorem may be stated as follows: G is a P-space if and only if, for any $x \in \beta G - G$, there exist Γ_α , U_α such that $\Gamma_\alpha \subset \bar{U}_\alpha \subset G$, Γ_α are closed, U_α are open, Γ_α and $P - U_\alpha$ are functionally separated, $\sum \Gamma_\alpha = G$, $\{U_\alpha\}$ is locally finite, $x \notin \sum \bar{U}_\alpha$. As a corollary, the existence of a complete uniformity on every paracompact space is obtained [cf. A. Dickinson, Amer. J. Math. 75 (1953), 224-228; MR 14, 1107]. The theorem of J. Dieudonné [C. R. Acad. Sci. Paris 209 (1939), 666-668; MR 1, 108] asserting that every c.r. space admitting of a one-to-one continuous mapping onto a metrizable space is a P-space, is generalized by putting "P-space with the first countability axiom" instead of "metrizable space". M. Katětov (Prague).

Smirnov, Yu. M. On completeness of proximity spaces. II. Trudy Moskov. Mat. Obsč. 4 (1955), 421-438. (Russian)

[For part I see same Trudy 3 (1954), 271-306; MR 16, 844.] Detailed proofs are given of theorems (with the

exception of the results concerning topological groups) contained in a previous note [Dokl. Akad. Nauk SSSR (N.S.) 91 (1953), 1281-1284; MR 16, 58]. M. Katětov.

Funayama, Nenosuke. Notes on lattice theory and its application. I. The lattice of all closed subsets of a T_0 -space. II. Combined space and its application. Bull. Yamagata Univ. (Nat. Sci.) 1, (1950) 91-100. (Japanese summary)

In the first part of this paper the author characterizes the lattices of closed sets of T , T_0 , and T_1 spaces as follows. The lattice of all closed sets of a T space is isomorphic with the lattice of all closed sets of some T_0 space (and of course, conversely). A complete, distributive lattice L is isomorphic with the lattice of all closed sets of a T space (and hence of a T_0 space) if and only if L is completely join-distributive, and, given $A < B$ in L , there exists C in L such that $A \leq C < B$, and C is join-irreducible in the quotient lattice A/B . A complete, distributive lattice L is isomorphic with the lattice of all closed sets of a T_1 space if and only if, given $A > B$ in L , there exists an atomic element p in L such that $p \leq A$ and $p \wedge B = 0$.

In the second part the author defines a compactification for T_1 spaces by means of maximal dual ideals in the lattice of all closed sets of the space. His method differs slightly from that of Wallman for the compactification of normal spaces. It is not clear what significance his method has for the more important and difficult problems of compactification for spaces satisfying stronger separation axioms than T_1 . There are some puzzling misprints.

O. Frink (University Park, Pa.).

Anderson, Frank W. A lattice characterization of completely regular G_δ -spaces. *Proc. Amer. Math. Soc.* 6 (1955), 757-65.

A topological space is called a G_δ -space if every one-point set is a G_δ . It is proved that a completely regular (c.r.) G_δ -space is characterized by its lattice (or by its ring) of real-valued continuous functions [an analogous result concerning c.r. spaces with the first countability axiom and rings of bounded continuous functions is well known, cf. E. Čech, *Ann. of Math.* (2) 38 (1937), 823-844].

It is to be noted that the corollary on p. 759 is erroneous; it is easy to see that $C_p(X)$ (the space of real-valued continuous functions on a c.r. X endowed with the topology of simple convergence on finite subsets) satisfies the first countability axiom if and only if X is countable (although $C_p(X)$ admits of a one-to-one continuous mapping onto a separable metric space whenever X is separable).

M. Katětov (Prague).

Zaidman, S., et Poenaru, V. De l'établissement d'une topologie pour certaines familles d'ensembles. *Com. Acad. R. P. Romine* 4 (1954), 195-197. (Romanian. Russian and French summaries)

Collins, Heron S. Completeness, full completeness, and k spaces. *Proc. Amer. Math. Soc.* 6 (1955), 832-835.

Let E be a completely regular space, and $C(E)$ the linear space of continuous, real-valued functions on E with the topology of uniform convergence on compact sets. It is known that the following implications hold, and are in general not reversible: $C(E)$ is fully complete $\Rightarrow E$ is a k -space $\Rightarrow C(E)$ is complete. [Full completeness is a concept developed by the author (*Trans. Amer. Math. Soc.* 79 (1955), 256-280; MR 16, 1030), and independently by Pták (*Čechoslovak Mat. Ž.* 3(78) (1953), 301-364; MR 16, 262) under the name of B -completeness; E is called a k -space if every subset, whose intersection with every compact set is closed, is closed.] This paper studies those E for which the above implications are reversible and shows, in particular, that they are if E is pseudofinite (i.e. every compact set is finite) or hemi-compact (i.e. $E = \bigcup_{i=1}^{\infty} C_i$, with each C_i compact, such that every compact CCE is a subset of some C_i).

E. A. Michael.

Wechsler, Martin T. Homeomorphism groups of certain topological spaces. *Ann. of Math.* (2) 62 (1955), 360-373.

The author considers the problem of characterizing a topological space F by its group of homeomorphisms $G(F)$. $G(F)$ is topologized by the point open topology and is not a topological group but does have the property that left and right translations are homeomorphisms of $G(F)$ onto itself.

Let \mathcal{F} denote the family of non-discrete Hausdorff spaces F satisfying the condition that for every integer $n \geq 1$ and for any two ordered subsets of F , each consisting of n distinct elements, there is a homeomorphism of F sending one ordered set into the other. The main theorem asserts that if F' and F belong to \mathcal{F} and if $G(F')$ and $G(F)$ are topologically isomorphic, then F' and F are homeomorphic. The proof follows from an intrinsic characterization of when a subgroup H of G is the group of all homeomorphisms of F leaving fixed some point of F .

E. Spanier (Chicago, Ill.).

Huhunaišvili, G. E. On a property of Uryson's universal metric space. *Dokl. Akad. Nauk SSSR (N.S.)* 101 (1955), 607-610. (Russian)

Uryson [Works on topology and other fields of mathematics, v. 2, Gostehizdat, Moscow-Leningrad, 1951, pp. 747-777; MR 14, 122] has given an example of a separable metric space U which contains an isometric copy of every separable metric space. Furthermore, U is homogeneous in the sense that if A and B are isometric finite subsets of U , there exists an isometry of U which interchanges A and B . Uryson showed the corresponding statement false in case A and B are uncountable, and raised the question of its truth in case A and B are countable. The author shows this false, even in case A and B are closed. However, he proves that if A and B are totally bounded isometric subsets of U , then there is an isometry of U interchanging A and B .

E. E. Floyd (Charlottesville, Va.).

Fadell, E. R. A property of compact absolute neighborhood retracts. *Duke Math. J.* 22 (1955), 179-184.

Theorem 1: Suppose X is a compact metric ANR, no component of which reduces to a single point, and f is a (continuous) mapping of X in itself. Then there exists a mapping g of X onto X , homotopic to f by a homotopy which for each t ($0 \leq t \leq 1$) has the same set of fixed points as f , providing that for each $x \in X$ there exists $x' \in X$ such that x and $f(x')$ lie in the same component of $X - \{x'\}$. This proviso is fulfilled automatically if X is connected and f has no fixed points, or if X is cyclically connected. The method of proof is to alter f by successive steps, each step involving only a small alteration of f on a small neighborhood in X ; the details could have been simplified by noting that each $x \in X$ has a (closed) neighborhood which is an AR. Some related theorems, mentioned without proof, are included in the following assertion. If f is a mapping of a connected normal T_1 space X with more than one point [the author requires X to be a Peano space, but this seems to the reviewer to be unnecessary] in a connected compact metric ANR space Y , and if X_0 is a proper closed subset of X , there exists a mapping g of X onto Y , which is homotopic to f rel. X_0 .

A. H. Stone.

Talmanov, A. D. On closed mappings. I. *Mat. Sb. N.S.* 36(78) (1955), 349-352. (Russian)

Suppose that X and Y are separable metric spaces and that f is a closed map of X onto Y . It is proved that if X is a Borel set, so also is Y . Also if X is a projective set, so also is Y .

E. E. Floyd (Charlottesville, Va.).

Halfar, Edwin. The isolated points of a set. *Portugal. Math.* 13 (1954), 125-128.

Let d be a derived-set operator on subsets of a set L and let j be the operator giving the isolated points of a set. The author defines, inductively, the operator d^* for ordinals α , and sets $J_\alpha = jd^*L$. He then proves numerous inclusion relations between these sets and their higher derived sets, for example: for all ordinals α and positive integers n , $J_{\alpha+n} \subset (jd)^n J_\alpha$.

M. E. Shanks.

Pesin, I. N. On the length of an everywhere discontinuous point set. *Uspehi Mat. Nauk (N.S.)* 10, no. 3(65) (1955), 153. (Russian)

A brief geometric proof that a certain Cantor set in the plane has infinite length.

E. E. Floyd.

Blumenthal, L. M., and Klee, V. L. On metric independence and linear independence. *Proc. Amer. Math. Soc.* 6 (1955), 732-734.

It is proved that a metric space (X) is homeomorphic with a linearly independent subset of the space $BC(X)$ of bounded continuous functions. The usual isometric imbedding (it would seem) does not do the trick.

R. Arens (Los Angeles, Calif.).

Fine, N. J., and Schweigert, G. E. On the group of homeomorphisms of an arc. *Ann. of Math.* (2) 62 (1955), 237-253.

The purpose of this paper is to study in detail the algebraic structure of the group of homeomorphisms G of the unit interval I . Denote by F the set of flows of I , where a flow is an element f of G leaving each end-point fixed. The following are among the theorems proved. (1) Every flow is a product of two translations, where a translation is an element of G leaving no interior point fixed. (2) Every element of G is a product of at most four involutions. (3) The center of G is the trivial subgroup $\{e\}$, while the commutator subgroup of G is F . (4) The normal subgroups of G are completely determined; they are G , F , Q , $\{e\}$, where Q is the set of all f whose fixed point set contains neighborhoods of each endpoint. (5) Every automorphism of G is inner. (6) Every element of the group of homeomorphisms of the circle is the product of at most three involutions of the circle. Other problems are also treated. For example, there is a necessary and sufficient condition that two elements of G be conjugate. There is also given a set of necessary and sufficient conditions that a given group be isomorphic to G .

E. E. Floyd (Charlottesville, Va.).

Eggleston, H. G. A property of plane homeomorphisms. *Fund. Math.* 42 (1955), 61-74.

Let E be the Euclidean plane and Θ the set of all homeomorphisms of E which leave either vertical lines invariant or horizontal lines invariant. Let Ξ be the set of homeomorphisms generated by finite superposition of members of Θ . S. Ulam inquired [*Fund. Math.* 24 (1935), 324] whether an arbitrary homeomorphism could be approximated by members of Ξ . The author answers this question in the negative if the approximation is to be uniform. However, homeomorphisms of a square, with sides parallel to the axes, which leave boundary points fixed can be approximated by members of Ξ leaving the boundary fixed. Hence the answer to Ulam's problem is in the affirmative if approximation is in the sense of pointwise convergence.

M. E. Shanks (Lafayette, Ind.).

Rodnyanskii, A. M. Differentiable mappings and the order of connectivity. *Mat. Sb. N.S.* 37(79) (1955), 69-82. (Russian)

The author denotes by R^n Euclidean n -space, by S^n its one-point compactification, by O and G open subsets of R^n , by Φ a (bi-)compact subset of G , by A an F_σ subset of G , and by f a differentiable mapping of G into R^n such that its Jacobian J does not take both signs and J is not identically zero in any interval. [To avoid appealing to unpublished results the author adds the hypothesis: (I) The partial derivatives of f are continuous.] The following theorems are established: (1) If $\text{Int}(A) = \emptyset$, then $\text{Int}(fA) = \emptyset$. (2) Each of the hypotheses (a) $G - \Phi$ has a bounded component whose closure is in G , (b) G is simply-connected and separated by Φ , (c) the number of components of $R^n - \Phi$ exceeds that of the components of $S^n - G$,

implies that $f\Phi$ separates R^n . Moreover, if $R^n - \Phi$ has an infinity of components then so has $R^n - f\Phi$. (3) If O is simply-connected and $f^{-1}O \neq \emptyset$, then $S^n - f^{-1}O$ has not fewer components than $S^n - G$. (4) If G is supposed a bounded simply-connected domain with frontier G_Φ and f has a continuous extension, denoted by the same letter, to $G + G_\Phi$, then $f^{-1}G_\Phi$ is a continuum and each component of the sets $R^n - fG_\Phi$ and $G - f^{-1}G_\Phi$ is simply connected. Moreover, in the case $n=2$, if O is a component of $G - f^{-1}G_\Phi$ in which $J \neq 0$, then f restricted to O is a topological mapping onto a component of $R^n - fG_\Phi$ and so has a differentiable inverse [with continuous partial derivatives if f is subject to (I)] whose Jacobian does not vanish and has the sign of J . L. C. Young (Madison, Wis.).

Dolcher, M. Geometria delle trasformazioni continue. Un teorema sulle trasformazioni di una corona circolare. *Riv. Mat. Univ. Parma* 5 (1954), 339-361.

Consider a circular corona in a plane bounded by circles α, β having like orientations. Let Φ be a continuous transformation from the closed corona into a plane, and consider a component A' of the complement of $\Phi(\alpha + \beta)$ for which the topological indices of any point in A' with respect to the images under Φ of α, β respectively have the same sign. Let n be the numerical difference between these topological indices. It is shown that the number of points in A' whose model sets contain fewer than n points is at most $n-1$. Precisely, for any set of k points p_i' in A' one has

$$\sum_{i=1}^k \{n - N[\Phi^{-1}(p_i')]\} \leq n-1,$$

where $N[\Phi^{-1}(p_i')]$ is the number of points in the model set $\Phi^{-1}(p_i')$. P. V. Reichelderfer (Columbus, Ohio).

Strother, Wayman. Fixed points, fixed sets, and M -retracts. *Duke Math. J.* 22 (1955), 551-556.

If X is a space, $S(X)$ denotes the space of non-null closed subsets of X , suitably topologized. Several results are obtained concerning $S(X)$ and its relation to X , such as the following. (1) If X, Y are compact Hausdorff spaces and $F: X \rightarrow S(Y)$ is continuous, then F is closed and induces $f: S(X) \rightarrow S(Y)$ (for $A = \overline{ACX}$ let $f(A) = \bigcup \{F(a) | a \in A\}$); moreover, f is continuous and, if $X=Y$, has a fixed element. (2) A retraction is devised from $S(E^2)$ onto $S(S^1)$, the existence of which, combined with well-known results, implies that $S(S^1)$ has the fixed-point property. (3) If X is a Peano space, then so is $S(X)$. [The reviewer suggests that in the study of multi-valued functions much clarity may be gained by using the notation $f: X \rightarrow 2^Y$ and $f: X \rightarrow S(Y)$ and reserving $f: X \rightarrow Y$ for single-valued functions.] S. Stein (Davis, Calif.).

Sikorski, R., and Zarankiewicz, K. On uniformization of functions. I. *Fund. Math.* 41 (1955), 339-344.

In this paper the authors prove theorems concerning mappings from the unit interval $[0, 1]$ into itself. One of them is: If f_1, \dots, f_n are mappings from the unit interval into itself which are monotone on every interval of a partition, where the partition depends only upon the mapping, then there are n mappings from $[0, 1]$ into itself such that $f_1\phi_1 = f_2\phi_2 = \dots = f_n\phi_n$. The other is a generalization of this theorem. This is similar to a theorem obtained by Homma in connection with a theorem on an arcwise connected subgroup of a vector group [Kôdai Math. Sem. Rep. 1952, 13-16; MR 14, 257].

H. Yamabe.

Sikorski, R. On uniformization of functions. II. *Fund. Math.* 41 (1955), 345-350.

As a more precise result about mappings from the unit interval $[0, 1]$ into itself than that obtained in part I reviewed above, the author shows that the set of points

$$\{(x_1, \dots, x_n); f_1(x_1) = f_2(x_2) = \dots = f_n(x_n);$$

$$x_1, \dots, x_n \in I; f_i \text{ mapping from } I \text{ into } I\}$$

connects the two points $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$ in I^n . The author also gives a condition on the f_i 's for the set to be locally connected. *H. Yamabe.*

Jaworowski, J. W. A theorem on antipodal sets on the n -sphere. *Bull. Acad. Polon. Sci. Cl. III.* 3 (1955), 247-250.

The author gave a condition in terms of homology for two closed antipodal sets on spheres, i.e. sets invariant under antipodal mapping, to intersect each other. Namely, if A is p -acyclic and B q -acyclic with $p+q > n-1$, then $A \cap B \neq \emptyset$. This result is obtained from a little more general result from which follows a condition for disjoint p -acyclic A and $(n-p-1)$ -acyclic B to be linked.

H. Yamabe (Minneapolis, Minn.).

Haman, K., et Kuratowski, K. Sur quelques propriétés des fonctions définies sur des continus univoques. *Bull. Acad. Polon. Sci. Cl. III.* 3 (1955), 243-246.

The authors prove the following theorems. (1) If C is a univoque, locally connected set admitting an involution $*$, then for any continuous real-valued function f on C , the set $\{p; f(p) = f(p^*)\}$ contains a connected antipodal set. (2) For any antipodal, closed, connected set F on the 2-sphere, there exists a continuous function f such that $\{p; f(p) = f(p^*)\} = F$, where $*$ is understood as the antipodal mapping. *H. Yamabe (Minneapolis, Minn.).*

Bourgin, D. G. On some separation and mapping theorems. *Comment. Math. Helv.* 29 (1955), 199-214.

The author proves a generalization of a theorem of Dyson's [*Ann. of Math.* (2) 54 (1951), 534-536; MR 13, 450] in a way similar to that of C. T. Yang [*ibid.* 60 (1954), 262-282; MR 16, 502]. The author's formulation is somewhat more combinatorial. His proof, as well as Yang's proof, deal with the properties of a compact sub-set in a cylindrical space which separates the upper bottom and the lower bottom. The author also remarks that when the antipodal mapping is replaced by a reflection in a hyperplane, similar results are obtained. The proofs for these results are also given by Yang [see the papers reviewed below]. *H. Yamabe (Minneapolis, Minn.).*

Yang, Chung-Tao. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. II. *Ann. of Math.* (2) 62 (1955), 271-283.

[For part I see *Ann. of Math.* (2) 60 (1954), 262-282; MR 16, 502.] The author proves theorems concerning mappings from spheres into euclidean spaces. Among them is a theorem which is an extension of a theorem of Dyson's [*Ann. of Math.* (2) 54 (1951), 534-536; MR 13, 450]. Making use of devices introduced by himself earlier, such as the B -index, he gives a proof of the following theorem. Let X be a compact set with an involution T having $B\text{-index} \geq mn + n - 1$, and let E and F be two closed subsets on $X \times X$ such that $E \cup F = X \times X$,

$$(x, y) \in E \leftrightarrow (y, x) \in E \leftrightarrow (Tx, y) \in F \leftrightarrow (x, Ty) \in F,$$

and $\{(x, x), x \in X\} \subset E = F$. Then for any mapping f on X

into m -dimensional euclidean space, there exist x_1, \dots, x_n on X such that

$$f(x_1) = f(x_2) = \dots = f(x_n) = f(Tx_1) = \dots = f(Tx_n),$$

with $(x_i, x_j) \in E \cap F$ for $i \neq j$, $1 \leq i, j \leq n$. Further generalizations are obtained by the author in the paper reviewed below. *H. Yamabe (Minneapolis, Minn.).*

Yang, Chung-Tao. Continuous functions from spheres to euclidean spaces. *Ann. of Math.* (2) 62 (1955), 284-292.

The author succeeds in proving a theorem which states that for any map from a mn -sphere into m -dimensional euclidean space, there are n diameters on the mn -sphere congruent to n given diameters which are mapped to a single point. This is an extension of theorems the author had proved in his preceding papers [see the preceding review and reference there]. The proof depends upon devices introduced by the author and upon lemmas which explore the relations between point-set topology and combinatorial topology. Connected with this problem is a conjecture by Knaster: Given a mapping from the n -sphere into m -dimensional euclidean space, do there exist $n-m+2$ points on the sphere mapped into a single point which are congruent to $n-m+2$ given points? *H. Yamabe (Minneapolis, Minn.).*

Dirac, G. A. Circuits in critical graphs. *Monatsh. Math.* 59 (1955), 178-187.

Let $L_k(n)$ be the largest integer such that every k -chromatic critical graph of order n contains a circuit of length $\geq L_k(n)$. For $k \geq 3$ the author proves that

$$\liminf \frac{L_{k+1}(n)}{\log^2 n} \leq k / \log^2 \left[k \left(\frac{k}{k-1} \right)^{k-1} \right],$$

thus generalizing the result of J. B. Kelly and L. M. Kelly for $k=4$ [*Amer. J. Math.* 76 (1954), 786-792; MR 16, 387]. *W. T. Tutte (Toronto, Ont.).*

See also: Rashevsky, p. 280.

Algebraic Topology

Weier, Josef. Über unwesentliche eindimensionale Singularitäten. *Monatsh. Math.* 59 (1955), 165-177.

Suppose that $F = (f^t)$ is a homotopy of a compact Euclidean polyhedron P in n -space R^n such that each f^t has only a finite number of fixed points. The author calls F finite in case the set A of all (p, t) with $p = f^t(p)$ is a homogeneous one-dimensional complex [*Monatsh. Math.* 59 (1955), 1-21; MR 16, 846]. If F is finite, a singularity of F is a certain type of open subarc of A ; in particular, such an arc which is open in A contains at most one (p, t) for each t , and is maximal with respect to these two properties. There are natural definitions of isolated singularity and of degree of a singularity. It is proved that if A' is a singularity of degree 0, then F can be closely approximated by a homotopy G whose corresponding singularity is isolated. Furthermore, if P is of dimension 2, and A' is a singularity of degree 0, then A' is inessential in the sense that F can be approximated by a G which has no singularity corresponding to A' . *E. E. Floyd.*

Toda, Hiroshi. Le produit de Whitehead et l'invariant de Hopf. *C. R. Acad. Sci. Paris* 241 (1955), 849-850.

In this note the author first gives a relation involving

the Whitehead product and the composition operation for elements belonging to various homotopy groups of spheres. He then gives the structure of the homotopy groups $\pi_{n+14}(S_n)$ and $\pi_{n+15}(S_n)$ for all n , asserting that the above mentioned relation plays a role in the calculations. As an example of the degree of complication of these results, we quote the following: $\pi_{18}(S_4)$ is the direct sum of cyclic groups of orders 1512, 2 and 30. $\pi_{23}(S_6)$ is the direct sum of a cyclic group of order 120 and five cyclic groups of order 2. $\pi_{n+15}(S_n)$ is the direct sum of a cyclic group of order 480 and a cyclic group of order 2 for $n > 16$.

As a corollary of these calculations, it follows that there does not exist any map of a 31-sphere onto a 16-sphere of Hopf-invariant one. Thus the first case in which the question of the existence of a map of Hopf-invariant one is still in doubt is the case of maps of a 63-sphere onto a 32-sphere.
W. S. Massey (Providence, R.I.).

Inoue, Yoshiro. On homotopy groups of function spaces. Proc. Japan Acad. 31 (1955), 222-227.

Let $(X; X_1, X_2, X_3)$ and $(Y; Y_1, Y_2, Y_3)$ be two tetrads such that X_1, X_2, X_3 are closed subspaces of X , and $X_3 \subset X_1 \cap X_2$, $Y_3 \subset Y_1 \cap Y_2$. Denote by Ω the space of maps $(X; X_1, X_2, X_3) \rightarrow (Y; Y_1, Y_2, Y_3)$ with the compact-open topology and by Λ the space of maps $(X_1, X_1 \cap X_2, X_3) \rightarrow (Y_1, Y_1 \cap Y_2, Y_3)$. Let $p: \Omega \rightarrow \Lambda$ denote the map defined by taking restrictions of the maps on X_1 . For each $f \in \Omega$, let $\Omega(f)$ denote the path-component of Ω containing f and $\Lambda(pf)$ the path-component of Λ containing pf . The author proves that $\Omega(f)$ is a fiber space over $\Lambda(pf)$ relative to p in the sense of Serre if some appropriate conditions are satisfied. Applications to various homotopy groups of function spaces are deduced. S. T. Hu (Athens, Ga.).

Chang, S. C. On certain polyhedron with the same homotopy groups as a given sphere. Acta Math. Sinica 4 (1954), 201-221. (Chinese. English summary)

In a previous paper [Sci. Sinica 3 (1954), 225-236; MR 16, 847], the author introduced a method to construct a map $[\phi_1, \phi_2, \phi_3]: S^{r_1+r_2+r_3-1} \rightarrow X$, if maps $f_i: S^{r_i} \rightarrow X$ ($i=1, 2, 3$) are given such that the Whitehead products $[f_i, f_j]$ are homotopic to zero in X . In the present paper, he has extended this operation to construct an element, $[\phi_1, \dots, \phi_n]$, of $\pi_{r_1+\dots+r_n-1}(X)$ if elements $\alpha_i \in \pi_{r_i}(X)$ ($i=1, \dots, n$), are given with appropriate properties. As a consequence, a special cell complex, $K_{r,n}$, is constructed which consists of one zero-dimensional cell, e^0 , one n -dimensional cell, e^n , such that $e^0 \cup e^n$ is an n -sphere, S^n , and one (kn) -dimensional cell if $1 < k \leq r$. Moreover, the (kn) -cell is attached by means of the map $[\phi_1, \dots, \phi_k]$ which is so introduced that each α_j ($j=1, \dots, k$) used in defining $[\phi_1, \dots, \phi_k]$ is the class determined by the identity map of the sphere S^n in $K_{r,n}$. The main theorem of this paper is

$$\pi_{n+1}(S^{n+1}) \approx \pi_n(K_{r,n}), \text{ if } n > 1, q \leq (r+1)n-2;$$

$$\pi_{n+1}(S^n) \approx \pi_n(K_{r,n}), \text{ if } 1 < q \leq 2r.$$

This sheds new light on the computation of the homotopy groups of spheres. By means of these methods, the author has successfully exhibited the kernel and the image of Freudenthal's suspension $E: \pi_n(S^n) \rightarrow \pi_{n+1}(S^{n+1})$.

S. T. Hu (Athens, Ga.).

Barratt, M. G. Track groups. I. Proc. London Math. Soc. (3) 5 (1955), 71-106.

In this paper, the author commences a detailed study

of "track groups". Let X be a space with base point x , and for every pair of spaces (P, Q) let $F(P, Q)$ be the function space of maps $f: (P, Q) \rightarrow (X, x)$. Then with rather mild restrictions on (P, Q) and X , the m -dimensional track group of (P, Q) relative to $(X, x; f)$ where $f \in F(P, Q)$ is the m -dimensional homotopy group of $F(P, Q)$ based at the point f . Since the space X and the base point x are fixed throughout most of the discussion, this group will usually be denoted by $(P, Q)^m$. Again with mild restrictions if (P, Q, R) is a triple, there is an exact sequence

$$\cdots \rightarrow (P, Q)^m \rightarrow (P, R)^m \rightarrow (Q, R)^m \\ \rightarrow (P, Q)^{m-1} \rightarrow \cdots \rightarrow (Q, R)^1.$$

Now let K be a finite complex, and denote its n -skeleton by K^n . Further, let π_m denote the m -dimensional homotopy group of X based at x , and let

$$\alpha: H^{n-1}(K^{n+1}, K^{n-2}; \pi_{m+n-1}) \rightarrow H^{n+1}(K^{n+1}, K^{n-2}; \pi_{m+n})$$

be the Steenrod square if $m+n > 3$, and the Pontrjagin square if $m+n=3$. The main theorem of the author may now be stated as follows: There are exact sequences of groups (possibly non-abelian)

$$0 \rightarrow A \rightarrow (K^{n+1}, K^{n-2}) \rightarrow C \rightarrow 0, \\ 0 \rightarrow H^n(K^{n+1}, K^{n-2}; \pi_{m+n}) \rightarrow C \rightarrow D \rightarrow 0,$$

where D is the subgroup of $H^{n-1}(K^{n+1}, K^{n-2}; \pi_{m+n-1})$ of elements u such that $\alpha(u)=0$, and A is

$$H^{n+1}(K^{n+1}, K^{n-2}; \pi_{m+n+1})$$

modulo the image of the Steenrod square: $H^{n-1}(K^{n+1}, K^{n-2}; \pi_{m+n}) \rightarrow H^{n+1}(K^{n+1}, K^{n-2}; \pi_{m+n+1})$. The extensions are abelian if $n=2$, and $m > 1$; and central otherwise.
J. C. Moore (Princeton, N.J.).

Wu, Wen-Tsun. Topological invariants of new type of finite polyhedrons. Acta Math. Sinica 3 (1953), 261-290. (Chinese. English summary)

The author describes a general method of deducing topological invariants of a given triangulable space which are in general not invariants of the homotopy type of the space.

Let K be a simplicial polyhedron and L a subpolyhedron of K such that (K, L) forms a normal pair. Then every simplex s of K which contains both points of L and points of $K-L$ is the join of a simplex s_0 in L and a simplex s_1 contained in $K-L$; hence every point x in the interior of s can be uniquely expressed in the form $(1-r)x_0 + rx_1$ where $x_0 \in s_0$, $x_1 \in s_1$ and $0 < r < 1$. The correspondence $x \rightarrow r$ extends to a continuous function $\lambda: K \rightarrow I$ such that $\lambda(L)=0$ and $\lambda(x)=1$ if x is in a simplex contained in $K-L$. For any given r with $0 < r < 1$, the author defines three subspaces $K_r^{(-)}(K, L)$, $K_r^{(0)}(K, L)$, $K_r^{(+)}(K, L)$ of K by the conditions $\lambda(x) \leq r$, $\lambda(x)=r$ and $\lambda(x) \geq r$ respectively. Obviously, L is a deformation of $K_r^{(-)}(K, L)$ and $K_r^{(+)}(K, L)$ is a deformation retract of $K-L$. In this way, the author obtains a collection of five subspaces of K , namely, $\{K, K_r^{(+)}(K, L), K_r^{(-)}(K, L), K_r^{(0)}(K, L), L\}$. The main theorem is that the homotopy type of this collection is a topological invariant of the pair (K, L) , i.e. it does not depend on the choice of the real number r and the triangulation of K .

Next, let L be any simplicial polyhedron and $K=L^n$ the n th topological power of L . Imbed L as the diagonal space of K . Then the author described a standard triangulation of K such that L becomes a subpolyhedron of

K and that (K, L) becomes a normal pair. Applying the main theorem to this particular normal pair (K, L) , one obtains a general method to deduce new topological invariants of L by taking the known invariants of the spaces $K_r^{(+)}(K, L)$, $K_r^{(-)}(K, L)$, $K_r^{(0)}(K, L)$ such as Euler-Poincaré characters, Betti numbers, etc.

By means of the standard triangulation of $K=L^n$, some reduction formulae for these new invariants are given. Examples are also given that these new invariants of L are not invariants of the homotopy type of L . Hence these new invariants are of finer character than the classical invariants and shed new light on the well-known unsolved problems such as the topological classification of closed 3-manifolds, etc. *S. T. Hu.*

Epifanov, G. V. On the density of two-dimensional polyhedra. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 189-190. (Russian)

Boltyanskii [same Dokl. (N.S.) 75 (1950), 605-608; MR 13, 149, 1139] defined the density of a space and showed that the density of a two-dimensional compactum is ≥ 6 . A. H. Stone [Proc. London Math. Soc. (3) 3 (1953), 338-349; MR 15, 336] showed the density of such a space to be ≤ 7 , and showed that most such spaces have density 7. He posed the problem of characterizing those 2-complexes P of density 6. The author states several lemmas and a theorem which gives such a characterization; such a P is either a pseudomanifold with non-empty boundary or one of a certain class of pseudomanifolds obtained by identifying points of a manifold. Local density is also defined; it is the minimum of the densities of closed neighborhoods of the point. It is stated that in order that a two-dimensional strongly connected polyhedron P be of density 6 at all of its points it is necessary and sufficient that P be a pseudomanifold (possibly with boundary).

E. E. Floyd (Charlottesville, Va.).

van der Waerden, B. L. Die Cohomologietheorie der Polyeder. Math. Ann. 130 (1955), 87-101.

This is an account of the cohomology theory based on functions of sets of points of the space. It is shown that this cohomology theory is the same for an arbitrary infinite polyhedron as the cohomology theory based on infinite cochains, thus generalizing a theorem of the reviewer proved for locally finite polyhedra [Ann. of Math. (2) 49 (1948), 407-427; MR 9, 523].

E. H. Spanier.

Lu, Chien-ke. Classification of 2-manifolds with a singular segment. J. Chinese Math. Soc. (N.S.) 1 (1951), 281-295. (Chinese summary)

The author considers the classification of compact, triangulable 2-manifolds having a set of singular points which is a closed segment. These objects may be defined precisely in the terminology of Seifert and Threlfall [Lehrbuch der Topologie, Teubner, Leipzig-Berlin, 1934, p. 48] as follows: They are pure 2-dimensional simplicial polyhedra P whose boundary is a 1-simplex σ , and are such that the complement of σ is an open 2-manifold. Using essentially elementary methods, the author obtains a theorem giving the topological classification of such objects. The final result is too lengthy to state here.

W. S. Massey (Providence, R.I.).

Schubert, Horst. Knoten und Vollringe. Acta Math. 90 (1953), 131-286.

This paper is concerned exclusively with the semi-linear

topology of spherical 3-space S ; all constructions are polyhedral and all mappings are simplicial.

An oriented simple closed curve in S is called a circuit (Knotenlinie). Two circuits are equivalent if one can be transformed into the other by an orientation-preserving autohomeomorphism of S . A knot (Knoten) is an equivalence class of circuits. The trivial knot (Kreis) is the class 1 that contains the plane circuits.

A closed normal neighborhood of a circuit l is a ring (Vollring) V , of which the circuit l is a core (Seele). The boundary T of a ring is a torus. A circuit m on T is a meridian (Meridian) if $m \cdot 0$ on T but $m \cdot 0$ in V . A meridian bounds a 2-cell f in the interior of V , and such a 2-cell is a meridian cell (Meridianfläche). A circuit b on T is a longitude (Breitenkreis) if $b \cdot 0$ on T but $b \cdot 0$ in $S-V$. A homeomorphic map of one ring on another is faithful if it carries meridian into meridian and longitude into longitude.

If a circuit k lies in V it is homologous in V to some multiple of the core l . The absolute value of this factor is the winding number (Umlaufzahl) $u = u_V(k)$ of k in V ; thus $k \sim \pm u \cdot l$. Each meridian cell of V intersects k in a certain number ($\leq \infty$) of points. The minimum of this number over all meridian cells is a non-negative integer $o = o_V(k)$ called the order (Ordnung) of V with respect to k .

The author partially orders the non-trivial knots by defining $\lambda \leq \kappa$ to mean that there is a ring V that contains circuits l and k ; representative of λ and κ respectively, such that l is a core of V and $o_V(k) > 0$. If, in addition, λ is not trivial and k is not a core of V , then $\lambda < \kappa$; in this case λ is called a companion (Begleitknoten) of κ . A (non-trivial) knot that has no companions is called simple. A descending chain of knots is necessarily of finite length.

If κ is a product of the prime knots $\lambda_1, \dots, \lambda_n$, then every factor $\lambda_1 \cdots \lambda_n$ of $\kappa = \lambda_1 \cdots \lambda_n$ is a companion of κ , and every companion of every prime factor λ_i of κ is a companion of κ . Moreover, these are the only companions of κ . In particular, every simple knot is prime. If κ is a double of λ (Schlingknoten κ mit Diagonalknoten λ), then λ is a companion of κ (provided λ is non-trivial), every companion of λ is a companion of κ , and these are the only companions of κ . A double of the trivial knot is either trivial or simple (and hence prime); a double of a non-trivial knot is always prime. If κ is a cable knot (Schlauchknoten) with core (Träger) λ then λ is a companion of κ (provided λ is non-trivial), and λ and its companions are the only companions of κ . A torus knot (cable knot with trivial core) is either trivial or simple; a cable knot with non-trivial core is always prime.

It is known that a product knot is determined by its prime factors (counted with their multiplicities), and that a doubled knot is determined by its carrier (Diagonalknoten), the twist (Verdrillungszahl), and the self-intersection number (Eigenschnittzahl). The author proves the analogous fact about cable knots: the core λ , winding number α , and twist (Verschlingungszahl) δ form a complete system of invariants of the cable knot κ , provided that $\alpha < |\delta|$ if κ is trivial. It is known that the genus of a product is the sum of the genera, and that the genus of a non-trivial doubled knot is 1. The author proves the analogous fact about cable knots: $g(\kappa) = \frac{1}{2}(\alpha-1)(\delta-1) + \alpha g(\lambda)$ where g denotes genus.

Let ϕ be a faithful map of an unknotted ring V^* upon a ring V with non-trivial core l , and let Z^* be the closed braid of α strings consisting of V^* and a circuit k^* that runs α (> 1) times monotonically around V^* . Then $Z = \phi(Z^*)$ is called a cable braid (Schlauchzopf) with core

(Träger) l ; it consists of $V = \phi(V^*)$ and the circuit $k = \phi(k^*)$. No knot can be represented in more than one way as a cable braid of a given number α of strings. (This is not true of closed braids.) If κ can be represented as a cable braid, then the cores of these cable braids can be arranged in a finite sequence $\lambda_1, \lambda_2, \dots, \lambda_m$ such that, for each index $i = 1, 2, \dots, m-1$, λ_i can be represented by a cable braid with core λ_{i+1} . Each λ_i is a companion of κ and every other companion of κ is a companion of λ_m . A cable braid is always prime.

If one generalizes the concept of a braid by dropping the requirement of monotonicity and replacing it by the weaker requirement that no individual string be knotted, one is led to the concepts of generalized closed braid and generalized cable braids. The above-mentioned theorems about cable braids apply also to generalized cable braids provided that one allows only those companions λ of κ for which the winding number is not zero.

There is a considerable amount of preparatory material about the semi-linear geometry of 3-space. The most important new result here would seem to be the following: If V and V^* are two non-trivially knotted rings and k is a circuit contained in the interior of each of them in such a way that $ov(k) > 0$, $ov^*(k) > 0$, then, by a semi-linear autohomeomorphism of space that leaves k fixed, it can be arranged that either (1) V^* is contained in the interior of V , or (2) V^* contains V in its interior, or (3) V^* contains the closed complement of V in its interior, or (4) there is a ring W that contains k in its interior and is contained in the interiors of both V and V^* in such a way that $ov(W) = ov^*(W) = 1$. In this last case a core of W represents the product of the knots represented by cores of V and V^* , and these factors are not prime factors.

R. H. Fox (Princeton, N.J.).

Schubert, Horst. Über eine numerische Knoteninvariante. Math. Z. 61 (1954), 245-288.

In connection with the paper reviewed above the

question arises whether a knot has only a finite number of companions. The object of this paper is to show that the answer is affirmative. Moreover, each comparison occurs in only a finite number of orders and in each order it has only a finite "multiplicity." This multiplicity is defined in a natural way that generalizes the multiplicity of a prime factor in a factorization of a knot. The principal tool is a numerical invariant b that has the property that $b(\lambda) < b(\kappa)$ whenever $\lambda < \kappa$.

The invariant b may be defined as follows: in each normed regular projection of k there is a minimal number of overpasses (Brücken); this is the number of maximal arcs that contain over crossing points but no under-crossing points [cf. Torres and Fox, Ann. of Math. (2) 59 (1954), 211-218, § 2; MR 15, 979]; the minimum of this number over all such projections is $b(\kappa)$. The number $b(\kappa)$ is clearly not greater than the minimum number $d(\kappa)$ of crossing-points; it is worth noting that it can actually be smaller than $d(\kappa)$ even for alternating knots (for the overhand knot $d=3$ but $b=2$).

A number of properties of b are developed: Theorem 1: If κ is non-trivial then $b(\kappa) \geq 2$. Theorem 3: If λ is a companion of k of order α , then $\alpha b(\lambda) \leq b(\kappa)$; moreover $b(\lambda) < b(\kappa)$. (This is a special case of theorem 2, which is a bit complicated to state). Theorem 4: If λ is a companion of κ , then its order is $\leq \frac{1}{2} b(\kappa)$. Theorem 5: If $b(\kappa) = 2$, then κ is simple. Theorem 6: A knot has only a finite number of companions, etc. Theorem 7: If $\kappa = \lambda \cdot \mu$, then $b(\lambda) + b(\mu) - 1 = b(\kappa)$. Theorem 8: If κ is a double of λ and is not trivial, then $b(\kappa) = 2b(\lambda)$. Theorem 9: If κ is represented by a cable braid with core λ and α strings, then $b(\kappa) = \alpha b(\lambda)$; if κ is represented by a closed braid with α strings, then $b(\kappa) \leq \alpha$. Theorem 10: If κ is a torus knot with winding number α and twist δ , where $\alpha < |\delta|$, then $b(\kappa) = \alpha$.

R. H. Fox (Princeton, N.J.).

See also: Kuranishi, p. 237.

GEOMETRY

Hebroni, Pessach. On tetrahedra with congruent faces. Riveon Lematematika 9 (1955), 25-28. (Hebrew. English summary)

Stockton, F. G. A set of triply perspective triangles associated with projective triads. Amer. Math. Monthly 62 (1955), no. 7, part II, 41-51.

Two triangles, $A = A_1 A_2 A_3$ and $B = B_1 B_2 B_3$, are said to be triply perspective if $A_1 A_2 A_3$ is in perspective, from centres C_1, C_2, C_3 , with $B_1 B_2 B_3, B_2 B_3 B_1, B_3 B_1 B_2$ [Veblen and Young, Projective geometry, vol. 1, Ginn, Boston, 1910, p. 100]. The centres C_i form a third triangle $C = C_1 C_2 C_3$, and the three triangles together form a Pappus configuration 9_3 . Dually, the three axes of perspective of A and C (or of A and B) form a new triangle B' (or C''), and so on. The author shows that the resulting infinite system of triangles can be represented by the vertices of the regular tessellation $\{3, 6\}$ of equilateral triangles filling the Euclidean plane. Each face of the tessellation represents a Pappus configuration. H. S. M. Coxeter.

Szász, Pál. On the rectification of circles. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 145-148. (Hungarian) Hungarian version of Acta Math. Acad. Sci. Hungar. 4 (1953), 251-253; MR 15, 645.

Kroch, A. Orthogonal axes in a pair of affine spaces. Technion. Israel Inst. Tech. Sci. Publ. 6 (1954/5), 81-83. (Hebrew summary)

The author considers a general affine transformation operating in Euclidean space. If a point A_1 is transformed into A_2 , one can always find three orthogonal lines through A_1 whose corresponding lines through A_2 are likewise orthogonal. For, a sphere with centre A_1 yields an ellipsoid with centre A_2 , whose three axes come from three diameters of the sphere which are conjugate and therefore perpendicular.

H. S. M. Coxeter.

Vişa, Eugen. Le théorème de Modénov. Gaz. Mat. Fiz. Ser. A. 7 (1955), 425-427. (Romanian. Russian and French summaries)

Bell, P. O. Generalized theorems of Desargues for n -dimensional projective space. Proc. Amer. Math. Soc. 6 (1955), 675-681.

Ein n -tupel linear unabhängiger Punkte wird als Simplex bezeichnet. Zwei Simplexe $A_1 \dots A_n$ und $A'_1 \dots A'_n$ heißen perspektiv, wenn es einen Punkt Z gibt, so dass Z, A_i, A'_i drei kollineare Punkte sind für jedes $i = 1, \dots, n$. Die beiden Simplexe können $k \geq 0$ Ecken gemeinsam haben. Durch Einführung homogener Koordinaten wird bewiesen: Wenn zwei Simplexe $A_1 \dots A_n$ und

$A_1 \dots A_n$ perspektiv sind und wenn $A_i \neq A_j$ ist für $k < j \leq n$, so existiert für jedes Paar i, j mit $k < i < j \leq n$ ein Punkt P_{ij} als Schnittpunkt von $A_i A_j$ mit $A_i' A_j'$, der linear abhängig ist von den $j-i$ Punkten $P_{ij}, P_{i+1, j}, \dots, P_{j-1, j}$, wo $P_k = P_{k, k+1}$ gesetzt ist. Die Umkehrung dieses Satzes lautet: Die Simplexe $A_1 \dots A_n$ und $A_1' \dots A_n'$ mit $A_i \neq A_i'$ für $j > k \geq 0$ mögen so liegen, dass für jedes Paar i, j ($i \neq j, i, j = 1, \dots, n$) ein gemeinsamer Punkt von $A_i A_j$ und $A_i' A_j'$ existiert. Ist $k \leq n-1$ und ist für ein bestimmtes Paar r, s aus $k < r < s-1 \leq n-1$ der Punkt P_{rs} linear abhängig von den Punkten $P_{rs}, P_{r+1, s}, \dots, P_{s-1, s}$, so sind die Simplexe perspektiv. Der Fall $k \geq n-2$ ist trivial. Aus diesen Sätzen werden Folgerungen gezogen und noch einige weitere Aussagen abgeleitet, die für die perspektive Lage von Simplexen hinreichend sind. *R. Moufang.*

Karzel, Helmut. Ordnungsfunktionen in nichtdesargueschen projektiven Geometrien. Math. Z. 62 (1955), 268-291.

In a plane an order function $h(A)$ is defined for every line h , and point A . In every case $h(A) = 0, +1$, or -1 , with $h(A) = 0$ if and only if A lies on h . The author studies order functions satisfying the line relation: If A, B, C are points of a line l , and h and k are two other lines through C , then $h(A)h(B)h(C)h(B) = +1$. For four points A, B, C, D on a line l , and further lines h , through D , g through C , k through A , and l through B , he proves $g(A)g(B)h(A)h(B) = k(C)k(D)l(C)l(D)$ and denotes this common value by the "separation symbol" $[C, D, A, B] = [g, h, A, B]$. The value of the separation symbol is the same for all harmonic sets.

The major part of the paper is taken up with a discussion of the relation of the order functions to the coordinatization of the plane by ternary rings. The finite point (x, y) lies on the line joining (m) and $(b, 0)$ if and only if $x = (m, y, b)$. Here (m) is an infinite point and (m, y, b) is a ternary operation with the usual properties needed for a projective plane. Writing $(a, b, 0) = ab$ and $(a, e, b) = a + b$, where e is the right unit, addition and multiplication are defined. The ring has an order function $o(a)$ if $o(0) = 0$ and otherwise $o(a) = +1$ or -1 , satisfying $o(ab) = o(a)o(b)$. The main result asserts that the existence of order functions $h(A)$ satisfying the line condition is equivalent to the existence of order functions on the ternary ring satisfying $o((m, r+z, a) - (n, r+z, b)) = o(m-n) \cdot o(r)$ when $(m, z, a) = (n, z, b)$, and $o((mr, a) - (m, r, b)) = o(a-b)$. These are immediate if distributive laws hold, and do not otherwise depend on properties of addition. *Marshall Hall, Jr.*

★ **Rozenfel'd, B. A.** Neevklidovy geometrii. [Non-Euclidean geometries.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 744 pp. 26.75 rubles.

This is a veritable encyclopedia of the algebraic-group-theoretical aspect of non-Euclidean geometry and as such extremely useful. According to the preface the book is also intended for reading through, but this seems to the reviewer an impossible undertaking, in particular because the author excels in thoroughness rather than elegance. Formulae covering a quarter of a page are frequent, and there are some filling an entire page.

The book begins with an axiomatic introduction to Euclidean geometry as a finite-dimensional vector space with a positive definite inner product. It then passes to a coordinate space (x^1, \dots, x^n) with coordinate convergence as topology, but an inner product of the form

$$(x, x) = -(x^1)^2 - (x^2)^2 - \dots - (x^l)^2 + (x^{l+1})^2 + \dots + (x^{n-l})^2,$$

which is denoted by ${}^{l+1}R_n$, by 1R_n for $d=0$ and by R_n for $l=d=0$. For complex x^i notations like ${}^1R_n(i)$ are used. After a brief introduction to Lie groups we find a detailed discussion (about 60 pages) of motions, in particular of the rotations and involutonic motions of 1R_n and ${}^1R_n(i)$.

Chapter II first derives the trigonometry on a hypersphere in 1R_n , which is shown to be, topologically, the product of an R_l and a hypersphere in R_{n-l} . Identification of diametrically opposite points of a hypersphere in ${}^1R_{n+1}$ leads to the n -dimensional non-Euclidean geometry 1S_n ; in particular, S_n is an elliptic and 1S_n a hyperbolic space. After discussing the properties of S_2 and 1S_2 the motions of 1S_n are studied in detail (about 50 pages). The chapter ends with a history of non-Euclidean geometry, which is interesting because it includes the contributions of Omar Khayâm and of Nasir-ad-din at-Tûsî, an Azerbaidjan mathematician of the 13th century. The author is (throughout the book) very fair in priority questions, but takes at times a militantly materialistic point of view in motivating discoveries.

Chapter III deals with the relations of non-Euclidean to projective geometry, including the symplectic spaces. The central point is an exhaustive study of projectivities. It is shown that the projective geometry of P_n with the linear line complex as basic element becomes the geometry of 2S_n and that the group of symplectic transformations of the three-dimensional symplectic space is isomorphic to the group of motions of 2S_3 . Chapter IV treats conformal geometry, in particular the relations between the points of ${}^{l+1}S_{n+1}$ and the hyperspheres of the conformal space 1C_n by means of Klein's polyspherical coordinates (a generalization of Darboux's pentaspherical coordinates). Of course, Poincaré's interpretation of non-Euclidean geometry appears in a duly general setting.

The fifth chapter deals with spinor representations of the motions of non-Euclidean spaces and contains much material which is due to the author. It contains in particular a detailed theory of alterions, i.e. real algebras 1A_n of rank 2^{n-1} whose base consists of a unit 1, of elements e_1, \dots, e_{n-1} and all products $e_{i_1} \dots e_{i_l}$ with $e_i e_j = -e_j e_i$ and $e_i^2 = -\varepsilon_i$, where $\varepsilon_i = \pm 1$ and l is the number of -1 's among the e_i . A complete list of the matrix algebras to which the various 1A_n are isomorphic is given. The spinor groups are defined as certain subgroups of alterion algebras, but their definition as well as their interpretation as groups of motions of non-Euclidean geometries are much too involved for a review. Special topics are the triality principle in seven-dimensional Euclidean spaces and their relation to the octaves (=Cayley numbers) and antioctaves.

The sixth chapter studies affine, projective, Euclidean, and non-Euclidean spaces over algebras other than the real numbers, in particular over the complex numbers, the dual numbers, the quaternions and anti-quaternions, again with such a wealth of details, that they cannot even be indicated here. It concludes with a discussion of the projective plane of the octaves and antioctaves.

The last chapter reports on, rather than treats in detail, the relations of the foregoing material to Riemannian geometry, first on the spaces of constant curvature, and then on the geometry of simple Lie groups and their relations to symmetric spaces. *H. Busemann.*

See also: Glagolev, p. 298.

Convex Domains, Extremal Problems, Integral Geometry, Distance Geometry

Hanani, H. On the number of lines and planes determined by d points. Technion. Israel Inst. Tech. Sci. Publ. 6 (1954/5), 58-63. (Hebrew summary)

Two theorems are proved, roughly to the effect that (1) d non-collinear points determine at least d lines, and (2) d non-coplanar points at least d planes. As noted by the author, (1) has been proved previously by de Bruijn and Erdős [Nederl. Akad. Wetensch., Proc. 51 (1948), 1277-1279; MR 10, 424] and by himself [Rivon Lematematika 5 (1951), 10-11; MR 13, 5]. The interest of (2) seems to be in the (three) exceptional cases where the d points determine exactly d planes; it is related to work by Motzkin [Trans. Amer. Math. Soc. 70 (1951), 451-464; MR 12, 849].

J. Riordan (New York, N.Y.).

Dvoretzky, Aryeh. A converse of Helly's theorem on convex sets. Pacific J. Math. 5 (1955), 345-350.

A family $\{K_\alpha\}$ of subsets of E^n is said to have the AH property provided whenever K_α' is a nonsingular affine image of K_α for each α , then either there is a point common to all the sets K_α' or some $n+1$ of these sets have an empty intersection. Now suppose $\{K_\alpha\}$ has at least $n+1$ members and each K_α is compact. From Helly's theorem it follows that if each K_α is convex, then $\{K_\alpha\}$ has the AH property; the author shows, conversely, that if $\{K_\alpha\}$ has the AH property and no K_α is contained in a hyperplane, then each K_α is convex. Also included are certain generalizations of this result, as well as examples showing the role of the various hypotheses. [Reviewer's note: In the proof of the Theorem, it is not sufficient that $Q_0^{(n)}$ be merely an extreme point of \hat{K}_i ; however, $Q_0^{(n)}$ will have the desired properties if it is a point of \hat{K}_i which is farthest from the origin.]

V. L. Klee, Jr.

Valentine, F. A. Three point arcwise convexity. Proc. Amer. Math. Soc. 6 (1955), 671-674.

An arc of E_2 is called convex provided it is contained in the boundary of a convex subset of E_2 , and a subset of E_2 is arcwise convex provided it contains, for each two of its points, a convex arc joining them. If each three points of S (SCE_2) are contained in a convex arc of S , then S is said to have the three point arcwise convexity property. The purpose of this paper is to characterize those closed subsets of E_2 (containing at least three points) with that property. It is established that any such set is either a closed convex set, or a convex curve (i.e., a closed connected part of the boundary of a plane convex set), or it is obtained by deleting from a closed convex set a bounded open convex subset.

L. M. Blumenthal.

Hemmi, Denzaburo. The minimum area of convex curves for given diameter and perimeter. Proc. Japan Acad. 30 (1954), 791-796.

Results are summarized from several previous papers [in particular: Bull. Yamagata Univ. (Nat. Sci.) 3 (1953), 1-11; 3 (1954), 55-76] on the shape and area of an oval with minimal area for given perimeter L and diameter D in the most difficult case when $3D < L < \pi D$.

W. Gustin (Bloomington, Ind.).

Hadwiger, Hugo, und Bieri, Hans. Eine Unstetigkeiterscheinung bei extremalen konvexen Rotationskörpern. Math. Nachr. 13 (1955), 19-24.

Let V denote the volume, F the surface, M the mean

curvature of a convex solid of revolution A . If $M^2 > \pi^2 F/2$, known inequalities are shown to imply $12VM \geq F^2$. The circular disc satisfies $V=0$, $M^2 = \pi^2 F/2$. Thus the absolute minimum of V for given M and F in the class of all A 's is discontinuous for $M^2 = \pi^2 F/2 \neq 0$. The authors note that this minimum is continuous for the A 's with $M^2 \leq \pi^2 F/2$.

P. Scherk (Saskatoon, Sask.).

Hadwiger, H. Konkave Eikörperfunktionale. Monatsh. Math. 59 (1955), 230-237.

Let A, B, \dots range through the bounded closed convex sets in euclidean k -space E^k . Let \mathfrak{M} [$\mathfrak{M} \subset \mathfrak{M}$] denote the class of all the functionals $\varphi(A)$ with the following properties. (i) $\varphi(A)$ is invariant with respect to translations [euclidean motions]. (ii) BCA implies $\varphi(B) \leq \varphi(A)$. (iii) $A_n \rightarrow A$ implies $\varphi(A_n) \rightarrow \varphi(A)$. (iv) $\varphi(\lambda A) = \lambda \varphi(A)$ for every $\lambda \geq 0$. (v) If $A \times B$ denotes the Minkowski sum of the non-void sets A and B , then $\varphi(A \times B) \geq \varphi(A) + \varphi(B)$. Thus \mathfrak{M} contains the width in a given direction, while the mean width $\beta(A)$, the thickness, the radius of the in-sphere, and the trivial functional which vanishes identically belong to \mathfrak{M} . If $\varphi \in \mathfrak{M}$ and if $\delta[d\delta]$ denotes rotation about the origin [the rotational density of integral geometry], $\varphi(A) = \text{Min}_\theta \varphi(A^\theta)$ and $\varphi(A) = \int \varphi(A^\theta) d\theta$ belong to \mathfrak{M} . If $\varphi \in \mathfrak{M}$, then $\varphi(A) \leq \varphi(K)$ for every A with $\beta(A) = \beta(K)$ [$K = \text{unit sphere}$].

Let $E^k \subset E^k$ denote an i -space subjected to translations orthogonally to itself and let dE^i denote the translation density of E^i . Let φ be a non-trivial functional in E^i satisfying (i)-(v); $\varphi > 0$. Then

$$\varphi(A) = \left\{ \int [\varphi(A \cap E^i)] dE^i \right\}^{1/(i+1-k)} \mathfrak{M}.$$

The case $\varphi = \text{length}$, yields Brunn-Minkowski's theorem that $V(A)^{1/k} \in \mathfrak{M}$ [$V = \text{volume}$]. Suppose for a given $\varphi \in \mathfrak{M}$ and a given B , the finite limit

$$\lim_{\lambda \rightarrow 0} [\varphi(A + \lambda B) - \varphi(A)]/\lambda$$

exists for every A . Then $\varphi_B(A) \geq \varphi(B)$. The case $B=K$, $\varphi(A) = V(A)^{1/k}$ yields the isoperimetric inequality.

P. Scherk (Saskatoon, Sask.).

Debrunner, Hans. Zu einem massgeometrischen Satz über Körper konstanter Breite. Math. Nachr. 13 (1955), 165-167.

Let K denote a convex body of constant width one in euclidean n -space. Let S be the unit sphere in that space and let \bar{K} denote the body symmetric to K with respect to the origin. Thus (1) $K + \bar{K} = S$ and (2) $\tau K + \sigma \bar{K} = \tau \bar{K} + \sigma K$. The mixed volumes V_k [W_k] of K and \bar{K} satisfy

$$(3) \quad V(\tau K + \sigma \bar{K}) = \sum_{k=0}^n \binom{n}{k} V_{n-k} \tau^k \sigma^{n-k}$$

$$[(4) \quad V(e^{K+\sigma S}) = \sum_{k=0}^n \binom{n}{k} W_{n-k} e^k \sigma^{n-k}];$$

here $V(X) = \text{volume of } X$. By (1), $eK + \sigma S = (e + \sigma)K + \sigma \bar{K}$. Hence (3) with $\tau = e + \sigma$ and (4) yield

$$(5) \quad \binom{n}{k} W_{n-k} = \sum_{j=0}^n \binom{n}{j} \binom{n-k}{j} V_j.$$

By (2) and (3), $V_k = V_{n-k}$. Hence (5) implies

$$(6) \quad W_{n-k} = \sum_{j=0}^k (-1)^j \binom{k}{j} W_{n-j} \quad [k=0, 1, \dots, n].$$

The odd values of k determine $(n+1)/2$ linearly inde-

pendent identities on which the others depend. [For earlier proofs of (6) cf. Dinghas, *Rev. Math. Union Interbalkan.* 3 (1940), 17-20; MR 2, 261; Santaló, *Portugal. Math.* 5 (1946), 195-201; MR 9, 526.] P. Scherk.

Gohier, Simone. Sur les calottes convexes tangentes tout le long de leur bord à une sphère. *C. R. Acad. Sci. Paris* 241 (1955), 154-156.

Theorem: If two isometric convex surfaces S, \bar{S} of class C^3 homeomorphic to a disc ("calottes") are tangent to two spheres $\sigma, \bar{\sigma}$ along their boundaries B, \bar{B} respectively, then they are either congruent or symmetrical. The author proves first (applying the theorem of Gauss-Bonnet) that B and \bar{B} are congruent; then the theorem follows from known results of Grotemeyer and Pogorelov.

L. A. Santaló (Buenos Aires).

Aleksandrov, A. D., and Sen'kin, E. P. On the rigidity of convex surfaces. *Vestnik Leningrad. Univ.* 10 (1955), no. 8, 3-13. (Russian)

Since a surface with positive Gauss curvature is locally convex it is clear what is meant with inner and outer normal. A surface S of positive curvature is said to turn its inside (outside) towards a point $p \notin S$ if for every (interior) point x of S the ray from p through x forms with the outer (inner) normal of S at x an acute angle. Let F_i ($i=1, 2$) have positive curvature, turn its inside towards p_i and let a plane through p_i exist such that F_i lies on one side of this plane. If there is an intrinsically isometric mapping of F_1 on F_2 under which corresponding boundary points of F_1 and F_2 have equal distances from p_1 and p_2 respectively, then F_1 and F_2 are congruent. Notice that F_i is not assumed to be convex; for instance, a spherical zone traversed twice is admitted. (Examples show that the existence of the plane through p_i is necessary for the validity of the theorem.) If, in particular, F_1 and F_2 are closed surfaces and p_i is chosen in F_i , a new proof for the rigidity of surfaces with positive curvature results. Let F_i , $i=1, 2$, have positive curvature, turn its outside towards p_i and let a ray L_i with origin p_i exist such that the ray parallel to L_i with origin at a point $x_i \in F_i$ forms an acute angle with the ray $p_i x_i$ as well as with the outer normal of F_i at x_i . If the remaining assumptions are the same as those in the preceding theorem, F_1 and F_2 are congruent.

It is indicated how these results can be extended to surfaces in spaces with constant curvature and to surfaces with non-negative (instead of strictly positive) curvature.

H. Busemann (Los Angeles, Calif.).

Fejes Tóth, L. Annäherung von Kurven durch Kurvenbogenzüge. *Publ. Math. Debrecen* 3 (1954), 273-280 (1955).

Let the class Γ^p of plane curves consist of all curves whose curvature is, as a function of the arc length, a polynomial of degree at most p , where $p=-1$ for straight lines and $p=1$ for circles. Denote by $\eta(Q, K)$ the deviation or Hausdorff distance of Q and K . Let K be a given curve with natural equation $K=K(s)$ ($0 \leq s \leq L$), and $K^{(p+1)}$ the $(p+1)$ -st derivative of $K(s)$. For each $n \geq 1$ and p there is a "polygon" Q_n^p with at most n sides, each of which lies in Γ^p , such that $\eta(Q_n^p, K)$ is minimal and

$$\Delta = \lim_{n \rightarrow \infty} \eta(Q_n^p, K) = C_p \left(\int_0^L |K^{(p+1)}|^{1/(p+3)} ds \right)^{p+3},$$

with $C_p = (p+2)[2^{p+3}(p+3)!^{p+3}/(p+3)^{(p+3)(p+2)}]^{-1}$ for even p and $C_p = [2^{p+4}(p+3)!]^{-1}$ for odd p . Hölder's inequality

yields $\Delta \leq C_p L^{p+3} \int_0^L |K^{(p+1)}| ds$ with equality only for curves in Γ^{p+1} so that the curves in Γ^{p+1} are, in a sense, those which can be least well approximated by polygons with sides in Γ^p . The cases $p=-1, 0$ were previously treated by the author [*Bull. Amer. Math. Soc.* 54 (1948), 431-438; MR 9, 525].

H. Busemann.

Nasu, Yasuo. On almost complete and almost metric spaces. *Tensor (N.S.)* 5 (1955), 58-67.

The notion of an almost metric space was defined by Menger [*Fund. Math.* 25 (1935), 441-458]. The present paper strengthens the postulates for such a space by assuming that the distance function is positive definite. A sequence $\{p_i\}$ is a strong Cauchy sequence provided $d > 0, \epsilon > 0$ imply the existence of a natural number N such that p_i is joined to p_j by a rectifiable curve C with diameter less than d and length less than ϵ for every $i, j > N, j > i$. A space is almost complete provided every strong Cauchy sequence has a limit. The paper extends to almost complete, almost metric spaces certain theorems concerning shortest arcs that were established in more restricted environments (e.g., almost complete, metric spaces) by Hopf and Rinow [*Comment. Math. Helv.* 3 (1931), 209-225] and Myers [*Trans. Amer. Math. Soc.* 57 (1945), 217-227; MR 6, 217]. Sample theorem: if R is locally compact, almost complete, almost metric, and the "function of triangle $\Delta(x)$ " is bounded, then any two points of R that are joined by at least one rectifiable curve are joined by an arc of shortest length.

L. M. Blumenthal (Columbia, Mo.).

See also: San Juan, p. 228.

Differential Geometry

★ Julia, Gaston. Cours de géométrie infinitésimale. Cinquième fascicule. Géométrie infinitésimale. Deuxième partie: théorie des surfaces. 2ème éd. Gauthier-Villars, Paris, 1955. iv+145 pp. 2400 francs.

[For fascicules I-IV see MR 15, 352; 16, 512, 744, 619.] This volume contains the classical theory of surfaces developed in the classical way, with an occasional touch of tensor methods. We find the Dupin indicatrix, the theorem of Bonnet on the line integral of the geodesic curvature, lines of curvature, conjugate systems and asymptotic lines, the theory of line congruences with some interesting results on singular lines taken from a paper of the author's [*C.R. Acad. Sci. Paris* 184 (1927), 1520-1522] and the mapping of surfaces on each other. The book ends with a note on imaginary elements.

D. J. Struik.

Ostrowski, Alexander. Über Evoluten und Evolventen ebener Kurven. *Arch. Math.* 6 (1955), 170-179.

Let S denote a simple plane arc. The coordinates x, y of its points are functions of its arc length s . The curvature r^{-1} exists and is finite, positive, and strictly monotonic. The equations $\xi = x - r dy/ds$ and $\eta = y + r dx/ds$ then define the evolute Σ of S . The following results are usually proved under stronger assumptions. The normals of S are tangents of Σ . The length of a sub-arc of Σ is equal to the difference of the corresponding radii of curvature of S . The first of the author's two proofs consists in verifying

$$\lim_{s \rightarrow s_0} \left(\frac{\xi(s) - \xi(s_0)}{r(s) - r(s_0)} + \frac{dy}{ds} \right) = \lim_{s \rightarrow s_0} \left(\frac{\eta(s) - \eta(s_0)}{r(s) - r(s_0)} \right) - \frac{dx}{ds} = 0.$$

This proof is simpler than the author's second proof and H. Kneser's [Arch. Math. 5 (1954), 77-80; MR 15, 984]. A similar idea is the basis of a proof of the following converse. Suppose the arc Σ is differentiable with respect to its arc length σ and the angle between the tangent of Σ and the positive x -axis is a continuous strictly monotonic function of σ . Let ξ and η denote the coordinates of the points of Σ and choose the constant c such that $\sigma+c>0$ on Σ . Define the arc S through $X=\xi-(\sigma+c)d\xi/d\sigma$, $Y=\eta-(\sigma+c)d\eta/d\sigma$. Then S is twice continuously differentiable and has the evolute Σ . P. Scherk.

Mirodan, R. Au sujet de l'existence d'une courbe autoparallèle à travers deux points, dans un espace A_2 . Com. Acad. R. P. Române 2 (1952), 213-216. (Romanian. Russian and French summaries)

Mirodan, R. Au sujet des courbes auto-parallèles des espaces à connexion affine A_2 . Com. Acad. R. P. Române 2 (1952), 505-511. (Romanian. Russian and French summaries)

These papers discuss a reduced form $y''=ay'^2+b$ of the equations of the paths in a two-dimensional geometry. The first applies an existence theorem to the case $ab \leq 0$ and finds that every point pair is joined by at least one path. The second applies an existence theorem to the case $0 < ab$ and finds that only restricted point pairs are joined by paths. J. M. Thomas (Durham, N.C.).

Sauer, Robert. Differentialgeometrische Eigenschaften der Integralfächen linearer partieller Differentialgleichungen zweiter Ordnung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1954 (1955), 305-314.

Let S be a non-developable surface $z=f(x, y)$ which satisfies the nowhere parabolic equation $az_{xx}+2bz_{xy}+cz_{yy}$, where a, b, c are functions of x, y . The paper gives differential-geometric theorems relating equation and solution: (i) In the hyperbolic case, the characteristics $cdx^2-2bdxdy+ady^2=0$ form a conjugate system on S . (ii) In the elliptic case, the asymptotic lines on S satisfy $cdx_1dx_2-b(dx_1dy_2+dx_2dy_1)+ady_1dy_2=0$. (iii) Legendre transformation takes the equation into a quasi-linear equation and a conjugate or asymptotic system on S into a like system on the new solution. (iv) Projective transformation of x, y into x', y' gives an equation whose solution is $z'=zg^{-1}$, where g is the denominator of x', y' . J. M. Thomas.

Marcus, F. De la définition de stratifiabilité en un sens, d'un couple de congruences de droites. Com. Acad. R. P. Române 1 (1951), 57-59. (Romanian. Russian and French summaries)

L'auteur établit deux critères de stratifiabilité, simple et double, pour deux congruences rectilignes K et K' dont les rayons sont en correspondance biunivoque. Pour que le couple (K, K') soit stratifiable dans un sens, il faut et il suffit qu'il existe trois surfaces telles, qu'en un triple de points d'intersection de ces surfaces avec un rayon quelconque de l'une K' des deux congruences, les plans tangents contiennent le rayon homologue de K . Pour la stratifiabilité dans les deux sens il faut et il suffit que la condition précédente soit vérifiée pour trois surfaces coupant les rayons de l'une K' des deux congruences, et pour deux seulement coupant les rayons de l'autre K .

P. Vincensini (Marseille).

Rozet, O. Sur les suites de Laplace. Bull. Soc. Math. Belg. 1954, 35-45 (1955).

L'auteur évoque plusieurs questions de géométrie faisant intervenir les suites de Laplace. C'est d'abord la

caractérisation des congruences de droites dont les foyers sont des sommets successifs d'une suite périodique de période donnée; l'auteur qui a, par ailleurs [Bull. Soc. Roy. Sci. Liège 20 (1951), 471-477; MR 13, 774] étudié le cas de la période six, donne la relation qui unit linéairement cinq points consécutifs de la suite la plus générale. C'est, en second lieu, la suite de Laplace L formée à partir des images U et V sur l'hyperquadrique de Klein des tangentes asymptotiques à une surface donnée S et la suite J associée à une congruence W dont S est une nappe focale, et auxquelles M. Godeaux a consacré d'importants travaux [La théorie des surfaces et l'espace réglé, Hermann, Paris, 1934]. Enfin, dans l'espace euclidien à trois dimensions, on peut considérer la suite de Laplace (\mathcal{M}) obtenue à partir du réseau conjugué orthogonal formé par les lignes de courbure au point générique M d'une surface; si M_u et M_v sont les centres de courbure principaux en M , ces points sont des sommets successifs dans une suite (m) circonscrite à (\mathcal{M}) . L'auteur termine par une application aux congruences de sphères de Ribaucour. M. Decuyper (Lille).

Petkantschin, B. Isometrie zwischen zwei Mongeschen Flächen. Bulgar. Akad. Nauk. Izv. Mat. Inst. 1 (1954), no. 2, 155-170. (Bulgarian. Russian and German summaries)

Im komplexen dreidimensionalen Euklidischen Raum ist die Mongesche Fläche als die Menge der Punkte auf den Geraden von der allgemeinen isotropen Regelschar G definiert. Die Gerade $g(u)$, $u \in U$, der Regelschar G ist durch den Punkt $x(u)$ und den Vektor $p(u)$ bestimmt. Dabei setzen wir voraus, dass im betrachteten einfach zusammenhängenden Gebiet U die Ungleichung

$$I = \left(p \cdot \frac{dx}{du} \left(\frac{dp}{du} \cdot \frac{d^2x}{du^2} \right) - \left(\frac{dp}{du} \cdot \frac{dp}{du} \right) \left[\left(\frac{dp}{du} \cdot \frac{dx}{du} \right) + \left(p \cdot \frac{d^2x}{du^2} \right) \right] \right) \neq 0$$

gilt. (In den Klammern sind die Skalarprodukte der Vektoren.)

Autor führt in die Betrachtungen den invarianten Parameter

$$\sigma = \int_{u_0}^u I \left[\left(\frac{dp}{du} \cdot \frac{dp}{du} \right) \left(p \cdot \frac{dx}{du} \right) \right]^{-1} du,$$

den Zentralpunkt

$$z = x + \frac{p}{I} \left[\left(\frac{dp}{du} \cdot \frac{dp}{du} \right) \cdot \left(\frac{dx}{du} \cdot \frac{dx}{du} \right) + \left(\frac{dp}{du} \cdot \frac{dx}{du} \right) \left(p \cdot \frac{d^2x}{du^2} \right) - \left(p \cdot \frac{dx}{du} \right) \left(\frac{dp}{du} \cdot \frac{d^2x}{du^2} \right) \right],$$

und die Normierung $e = p/(p \cdot dz/d\sigma)$ ein.

Dann bekommt man die Invarianten von G

$$R = R_0 e^{-\sigma}, \quad S = \sqrt{\left(\frac{dz}{d\sigma} \cdot \frac{dz}{d\sigma} \right)}, \quad T = \left(\frac{de}{d\sigma} \cdot \frac{dz}{d\sigma} \right)$$

und die Frenetsche Formel für G in denen R, S, T auftreten.

Weiter sucht der Autor alle Isometrien von zwei Mongeschen Flächen F_1, F_2 und bekommt den folgenden Satz: Es sei M eine Isometrie der Fläche F_1 auf F_2 . Dann induziert sie eine Abbildung M_0 der Regelschar G_1 auf die Regelschar G_2 , bei welcher in den entsprechenden Geraden g_1, g_2 die Gleichungen

$$(a) \quad R_1^2 = R_2^2, \\ T_1^2 - 4T_1 + 2 \frac{dT_1}{d\sigma} - \frac{S_1^2}{R_1^2} = T_2^2 - 4T_2 + 2 \frac{dT_2}{d\sigma} - \frac{S_2^2}{R_2^2}$$

gelten. Die Isometrie M wird analytisch durch

$$(b) \quad \sigma_2 = \sigma_1, \quad v_2 = v_1 + R_1^2(T_1 - T_2)$$

bestimmt, wenn $g_1(0), g_2(0)$ sich in M entsprechen. Umgekehrt, eine den Bedingungen (a) genügende Abbildung M der Regelschar G_1 auf die Regelschar G_2 lässt sich durch (b) zu einer Isometrie von F_1 auf F_2 erweitern.
F. Vyěichlo (Prag).

Borisov, Yu. F. Geometry of a half-neighborhood of a curve in a two-dimensional manifold of bounded curvature. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 537-539. (Russian)

On a surface of bounded curvature (all concepts are to be understood in the sense of A. D. Alexandrov) consider a curve L homeomorphic to a segment whose geodesic curvature has bounded variation and is everywhere less than π (i.e. L has no cusps) and such that the (Gauss) curvature at the endpoints A, B of L is less than 2π . Let AA_1 and BB_1 be shortest arcs which have no other common points with L than A, B , finally R a geodesic polygon connecting A_1 to B_1 such that L, AA_1, R, B_1B bound on the surface a domain U homeomorphic to a disk. Denote by $L(\tau)$ the set of points in U with distance τ from L . Then a positive τ_0 exists such that for $\tau \leq \tau_0$ the set $L(\tau)$ is a rectifiable curve connecting a point A_τ of AA_1 to a point B_τ on BB_1 . If, beginning at A_τ , a parameter σ ($0 \leq \sigma \leq 1$) proportional to the arc length is introduced on $L(\tau)$, then $\sigma \rightarrow x, \tau \rightarrow y$ gives a topological mapping of the subset $\tau \leq \tau_0$ of U on a rectangle in the (x, y) -plane. After a suitable change of the parameter σ (except on L) a curve $\tau = f(\sigma)$, where $f(\sigma)$ satisfies a Lipschitz condition, is rectifiable and so is $\tau = \lambda f(\sigma)$ for $0 \leq \lambda \leq 1$. If $S(\lambda)$ is the length of this curve, then for $dS(\lambda)/d\lambda|_{\lambda=0}$ a formula similar to the usual holds. Also, if $\alpha(\lambda)$ is the area bounded by $\tau = \lambda f(\sigma)$, L and the corresponding arcs of AA_1 and BB_1 , then $d\alpha(\lambda)/d\lambda|_{\lambda=0} = S(0) \int_0^1 f(\sigma) d\sigma$. This yields a necessary condition analogous to the usual for solutions of the isoperimetric problem, in particular the constancy of the geodesic curvature, if no curve with geodesic curvature of bounded variation has negative Gauss curvature.
H. Busemann (Los Angeles, Calif.).

Gergely, Eugen. La classification des surfaces sur la base de leur géométrie intrinsèque. Acad. R. P. Romine. Fil. Cluj. Stud. Cerc. Şti. 5 (1954), 27-44. (Romanian. Russian and French summaries)

Gergely, Eugen. La classification des surfaces, basée sur la géométrie intrinsèque. Com. Acad. R. P. Romine 5 (1955), 27-30. (Romanian. Russian and French summaries)

Consider a surface S with an intrinsic metric in the sense of A. D. Alexandrov, i.e. a 2-dimensional manifold metrized so that the distance of two points equals the greatest lower bound of the lengths of the curves connecting them. A domain on S is convex if it contains with any two points at least one of their shortest connections, and is quasi-convex if each of its points has arbitrarily small convex neighborhoods. The complement of a quasi-convex domain is called a-convex and its boundary curve a separating curve. There is an at most denumerable number of separating curves. Characterizations of the system of separating curves from the topological point of view are given in this sense: if the sides of a separating curve are labelled according to where the quasi-convex and a-convex domain lies, then necessary and sufficient conditions are found under which a system of curves with

a labelling can be realized as system of separating curves on a surface with the prescribed labelling. A separating curve of finite length has common points with the boundaries of a finite number of quasi-convex and a-convex domains.
H. Busemann (Los Angeles, Calif.).

Šulikovskii, V. I. On infinitesimal bending of a surface. Uč. Zap. Kazansk. Univ. 114 (1954), no. 2, 79-87. (Russian)

Known results of the local theory of infinitesimal bending are expounded by tensor methods. The differential equation for the characteristic function of infinitesimal bending is given. There are obtained the 12 Darboux surfaces, corresponding pairwise either from the orthogonality of the linear elements or the parallelism of the tangent planes. Given are formulas for the characteristic functions of the surfaces connected with a bundle of Moutard transformations. Some results concerning W congruences are also derived.

E. G. Poznyak (RŽMat 1955, no. 5268).

Thomas, T. Y. Kinematically preferred co-ordinate systems. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 762-770.

In this paper, the author defines kinematically preferred coordinate systems, studies their properties, and shows how quantities may be constructed which are invariant under a group of moving rectangular coordinate transformations to which the preferred coordinate systems belong. First, a basic relation between the usual second-order tensor known as the "curl of the velocity" of two moving rectangular coordinate systems is determined. [In a communication to the reviewer, the author has remarked that differentiation with respect to \bar{x}_β not \bar{y}_β , as stated in the paper, furnishes his relation (4)]. This is not a tensor relation but involves time derivatives of the coordinate transformation coefficients. The kinematically preferred systems are defined as those having their origin at some moving point and such that the "curl of the velocity" vanishes at this origin. This leads to conditions on the coordinate transformation coefficients relating these preferred coordinate systems to arbitrary moving rectangular coordinate systems and to other preferred systems. As the author remarks, the basic idea is similar to that relating ordinary and normal coordinates in geometry — except there the symmetric part of the connection vanishes at the origin of normal coordinates; here the "curl of the velocity" vanishes at the origin of the kinematically preferred coordinates. In these preferred coordinate systems, total time derivatives of tensors are tensors — hence the importance of these coordinate systems. Relations for the total time derivative of tensors in arbitrary moving rectangular coordinate systems and formulas for the coordinate extension of tensors (the partial derivatives of tensors in the preferred systems) are found. Finally, the author summarizes some of his results in a replacement theorem which shows how to express an invariant of tensors in motionless rectangular coordinate systems in terms of the corresponding tensors of kinematically preferred coordinate systems.
N. Coburn.

★ **Coburn, Nathaniel.** Vector and tensor analysis. The Macmillan Company, New York, 1955. xii+341 pp.

This book has grown from courses on vector analysis and on compressible fluids, and turbulence at the senior-graduate and graduate level respectively at the University of Michigan.

Vector analysis is dealt with in Part I (108 pp.), containing algebra, differentiation and integration and applications to rigid body dynamics and fluid dynamics.

Tensor analysis is introduced in Part II (94 pp.) starting from vector analysis and cartesian coordinates and up to page 191 only for three-dimensional euclidean space. Only the theorems of Gauss and Stokes in the last chapter of this part are derived from the more general point of view of the euclidean n -dimensional space.

In the third part (111 pp.) the author first returns to three-dimensions for the discussion of the geometry of surfaces, but for the deduction of the curvature tensor a more general point of view is accepted. The last three chapters of this part deal with applications to the theory of elasticity and viscous fluids, of compressible fluids and of homogeneous statistical turbulence.

The book is very easy to read and will be welcomed by many students in physics and engineering.

J. A. Schouten (Epe).

Riemannian Geometry, Connections

Haimovici, Adolf. The notions of geometry of spaces with an affine connection. *Gaz. Mat. Fiz. Ser. A* 7 (1955), 344-359. (Romanian)
Expository paper.

Auslander, L. and Markus, L. Holonomy of flat affinely connected manifolds. *Ann. of Math.* (2) 62 (1955), 139-151.

In this review an affine connexion will be called "flat" if and only if the curvature and torsion are both zero. Let M be a C^∞ manifold. A family of coordinate systems covering M is called "affine" if the Jacobian of any two of them is a constant matrix on the components of the regions where their domains overlap. It is shown that such a covering gives rise to a unique flat affine connexion for which Γ_{jk}^i for these coordinate systems. Conversely, a flat affine connexion is always related in this way to an affine covering.

From now on let Γ denote a flat affine connexion on M . There is a natural homomorphism of the fundamental group with base point m onto the holonomy group H at m . If M is simply connected then Γ is a Riemannian connexion in the Riemannian metric obtained by taking any scalar product at m and parallel translating it to other points. Thus the various well known facts about such Riemannian manifolds hold on the assumption that M has a flat affine connexion.

Let B be the bundle of bases over M . For any $b \in B$ there is a unique "horizontal" submanifold of B thru b , and this manifold covers M under the projection of B onto M . The group of covering transformations is shown to be the holonomy group; this covering space has zero holonomy group, and is shown to be minimal with respect to this property. W. Ambrose (Cambridge, Mass.).

Evtušik, L. E. On the geometry of a double integral. *Mat. Sb. N.S.* 37(79) (1955), 197-208. (Russian)

In this paper geometrical objects are studied which are connected with the integral

$$\iint F(x^i, x^j, \frac{\partial x^j}{\partial x^i}, \frac{\partial^2 x^j}{\partial x^i \partial x^k}) [dx^1 dx^2] \quad (I, J=1, 2, 3; i, j, k=1, 2)$$

invariantly under the infinite group of analytical transfor-

mations of three variables x^1, x^2, x^3 . The same ideas and methods are used as in the paper by Aussem [Dokl. Akad. Nauk SSSR (N.S.) 85 (1952), 253-255; MR 14, 320] dealing with the case of the integral

$$\iint F(x^i, x^j, \partial x^j / \partial x^i) [dx^1 dx^2].$$

Making use of Cartan's theory of invariant differential forms, the author calculates the objects of first, second and third orders defined on the base space $(x^1, \bar{p}_i, \bar{a}_{ij}, \bar{a}_{ijk})$, i.e. third-order space of dimension 12, and deduces the invariant directions $b^i e_i + e_3$ and the metric tensor g_{IJ} at any element of the base space. Applying the theory, the variations of the integral and Euler's equations are expressed in invariant forms. At the end the theory is compared with that of Aussem cited above.

A. Kawaguchi (Sapporo).

Kilmister, C. W. The application of certain linear quaternion functions of quaternions to tensor analysis. *Proc. Roy. Irish Acad. Sect. A* 57 (1955), 37-52.

This paper is written in two parts. In the first one will be found a collection of results on linear functions of quaternions including singular ones. In addition, the connection of quaternions with special relativity is described. In the second part of the paper results of Whittaker and Ruse on self-dual anti-symmetric tensors in Minkowski space are derived and generalised. The connection between quaternions and the two component spinor calculus is discussed.

A. H. Taub (Urbana, Ill.).

See also: Zalcov, p. 330.

Algebraic Geometry

Godeaux, Lucien. Sur certaines courbes tracées sur une surface multiple. *Bull. Soc. Roy. Sci. Liège* 24 (1955), 201-208.

Godeaux, Lucien. Addition à la note sur l'ordre d'une involution cyclique appartenant à une surface algébrique. *Bull. Soc. Roy. Sci. Liège* 24 (1955), 209-211.

★ **Mandzyuk, A. I.** Application of a special addition of points of a curve C^3 of the third order to the proof of a theorem on closed $2n$ -gons. *Nomografičeskii sbornik* [Nomographic collection], pp. 39-45. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Everything in this paper is explicitly or implicitly contained in Section 24 of van der Waerden's *Einführung in die algebraische Geometrie* [Springer, Berlin, 1939]. The theorem to which the title refers is the following: If P_0 and Q are two fixed points of a third-order plane curve C , and A_1 is any point of C , let B_1 be the third intersection of the line QA_1 with C , then A_2 the third intersection of $P_0 B_1$ with C ; proceed with A_2 as with A_1 , etc., obtaining successively $B_2, A_3, B_4, A_5, \dots$. If $A_{n+1} = A_1$ for some n , then $A_{n+1} = A_1$ for any position of A_1 on C with the same P_0, Q , and n . H. Busemann.

★ **Glagolev, A. A.** On a set of triangles. *Nomografičeskii sbornik* [Nomographic collection], pp. 25-34. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

In P^n a hypersurface $\sum a_{i_1, \dots, i_n} x_{i_1} \dots x_{i_n} = 0$ of order n and hypersurface $\sum \alpha_{i_1, \dots, i_n} x_{i_1} \dots x_{i_n}$ of class n are apolar if $\sum a_{i_1, \dots, i_n} \alpha_{i_1, \dots, i_n} = 0$. Synthetic interpretations of apolarity

have been known only for $n=2, r=1$; $n=2, r=2$; $n=2, r=3$; and in special cases for $n=3, r=2$. The present paper supplies an interpretation for $n=3, r=1$ and simplifies some work of A. K. Vlasov by using triangle sets I_D^2 defined as follows: On a plane conic C^2 consider a point triple ABC and let $D \notin C^2$. Then I_D^2 consists of all triangles circumscribed to C^2 and inscribed to a conic of the pencil of conics with base points $ABCD$. The notation I_D^2 is justified by the fact that any other triple $A'B'C'$ in I_D^2 leads to the same set I_D^2 . There is exactly one triple in I_D^2 whose Brianchon point is D . A cubical polar system S_3^2 of Vlasov consists of ∞^2 point triples on a conic C^2 such that for any point X on C^2 the pairs YZ for which $XYZ \in S_3^2$ form an involution whose center lies on a fixed line. The points of contact of the sides of the triangles in an I_D^2 form a S_3^2 . The above mentioned interpretation of an apolarity for $n=3, r=1$ is this: two triples $XYZ, X'Y'Z'$ on a conic (and, by projection from a point on the conic, on a line) are apolar if and only if the vertices of the triangles $ABC, A'B'C'$ formed by the tangents of C^2 at $XYZ, X'Y'Z'$, respectively, and their Brianchon points D, D' lie on a conic.

H. Busemann.

Semple, J. G., and Kirby, D. Local dilatation. J. London Math. Soc. 30 (1955), 417-422.

In a recent paper B. Segre developed, by arguments that were mainly geometrical, the principal properties of dilatations in the algebraic domain [Ann. Mat. Pura Appl. (4) 33 (1952), 5-48; MR 14, 683]. The same author also discussed in a subsequent paper the properties of dilatations in the analytic domain, among other things stating without proof that, if a submanifold M_δ of an analytic variety V_δ is "blown-up" into a submanifold $M'_{\delta-1}$ of a corresponding variety V'_δ by means of a dilatation ($0 \leq \delta \leq d-2$), and if O, O' are any two corresponding points of $M_\delta, M'_{\delta-1}$, then it is always possible to choose local coordinate systems (x_1, \dots, x_δ) on V_δ at O and (y_1, \dots, y_δ) on V'_δ at O' such that the equations of the dilatation in the neighbourhood of the pair (O, O') reduce to the form

$$(1) \quad x_i = y_i \quad (i=1, \dots, \delta); \quad x_j = y_j y_\delta \quad (j=\delta+1, \dots, d-1); \\ x_d = y_\delta$$

[cf. equations (4) in B. Segre, Rend. Circ. Mat. Palermo (2) 1 (1953), 373-379; MR 15, 351].

In the present paper this result is obtained — in the algebraic domain — as a consequence of a theorem giving necessary and sufficient conditions in order that, by suitably chosen local parametrizations, the relations (1) express in the neighbourhood of a pair (O, O') the correspondence between two algebraic varieties V_δ, V'_δ , when V'_δ is the transform of V_δ by means of a linear system of primals passing simply through an irreducible algebraic M_δ of V_δ , O is a simple point of M_δ , and O' is one of the transforms of O .

B. Segre (Rome).

Roth, Leonard. Some properties of pseudo-Abelian varieties. Ann. Mat. Pura Appl. (4) 38 (1955), 281-302.

A pseudo-Abelian variety W_p of type q ($1 \leq q \leq p-1$) is a non-singular variety admitting a permutable continuous group \mathfrak{G} of ∞^q automorphisms whose trajectories form a congruence $\{V_q\}$, whose generic member is a Picard variety on which \mathfrak{G} acts transitively and without exception. Such a variety contains a second congruence of ∞^q varieties $\{V_{p-q}\}$, which are all birationally equivalent, and the variety representing this congruence is either a Picard or an Abelian variety. The properties of W_p are

studied by deriving a mapping of W_p on the d -ple variety W_p^* , where W_p^* is the product of the varieties V_q^* and V_{p-q}^* , which are in regular birational correspondence with the congruences $\{V_q\}$ and $\{V_{p-q}\}$, and where $d = (V_q, V_{p-q})$. Among many other results of interest the following theorems are proved. (1) If $g_i(W_p)$ is the number of linearly independent i -ple integrals of the first kind on W_p , then $g_i(W_p) \geq g_i(V_q^*) + g_i(V_{p-q}^*)$. (2) The canonical systems $X_k(W_p)$ are of order zero for $k < q$. (3) The canonical invariants of W_p (i.e. the orders of the various series of sets of points defined by intersections of canonical varieties) are all zero. A number of special classes of pseudo-Abelian varieties are studied, including improperly Abelian varieties (mapped by superficially irregular involutions in Picard varieties), and types which can be mapped on multiple quasi-Abelian varieties.

J. A. Todd (Cambridge, England).

Vesentini, Edoardo. Ancora sulle classi caratteristiche e sulle varietà covarianti d'immersione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17 (1954), 196-203.

It is observed that in the author's earlier proof [same Rend. (8) 16 (1954), 199-204; MR 16, 66] of the identity of the successive covariants of immersion of a non-singular irreducible algebraic variety imbedded in another variety V and the characteristic classes of the tangent bundle of P itself the assumption is made of the existence of an irreducible non-singular hypersurface passing through P and belonging to V . This paper sketches a proof which avoids this assumption. It also gives a proof of a formula of Gamkrelidze [Dokl. Akad. Nauk SSSR (N.S.) 90 (1953), 719-722; MR 15, 459] on the characteristic classes of an algebraic variety in a projective space.

S. Chern (Chicago, Ill.).

Gherardelli, Francesco. Un'osservazione sulla catena delle sizigie di un ideale di funzioni theta. Boll. Un. Mat. Ital. (3) 10 (1955), 190-194.

Andreotti [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° 27 (1952), no. 4; MR 15, 344] has shown that certain graded rings of theta functions are Noetherian and satisfy the Hilbert theorem on chains of syzygies. The author gives alternate proofs of these facts.

M. Rosenlicht.

Nishi, Mico, and Nakai, Yoshikazu. On the hypersurface sections of algebraic varieties embedded in a projective space. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 29 (1955), 1-5.

Let V be a variety in a projective space L of dimension N , and k a field of definition of V . Let g be an integer > 0 . Denote by $l(m)$ the dimension of the family of hypersurfaces (positive cycles of dimension $N-1$) of degree m in L . The authors prove that, if m is large enough, and if H is a hypersurface of degree m whose Chow point is of transcendence degree $\geq l(m) - g$ over k , then $V \cdot H$ is defined and irreducible. The proof is based on the consideration of those specialisations of H which are of the form $H_{m-1} + H_0$, with some fixed given H_{m-1} (for a suitable choice of l) and on a lemma to the effect that, for any given r ($1 \leq r \leq N-1$) and d , and for sufficiently large m , the carrier of H cannot contain any r -dimensional variety of degree $\leq d$. Let $s(m)$ be the dimension of the linear system Λ cut out from V by the hypersurfaces of degree m ; the author considers as evident the fact that the elements of Λ which are reducible form a bunch B of subvarieties of Λ . This being granted, it follows im-

mediately that, if $s'(m)$ is the dimension of B , then $s(m) - s'(m)$ increases indefinitely with m .

In the statement of Lemma 2, p. 2, the words "of dimension r " should be inserted between "subvarieties" and "of degree not greater than d ". C. Chevalley.

Samuel, Pierre. Simple subvarieties of analytic varieties.

Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 647-650.

Let k be a field, $A = k[[X_1, \dots, X_n]]$ the formal power series ring in n indeterminates, \mathfrak{pCp} prime ideals in A . The analytic (i.e. algebroid) variety associated with \mathfrak{p} is said to be a simple subvariety of the analytic variety associated with \mathfrak{p} if the local ring $(A/\mathfrak{p})_{(\mathfrak{p}/\mathfrak{p})}$ is regular. It is shown (by an analogous proof) that if k has characteristic $p \neq 0$, then Zariski's generalized jacobian criterion for the simplicity of algebraic subvarieties of algebraic varieties can be extended to the analytic case. M. Rosenlicht.

★ Samuel, P. Méthodes d'algèbre abstraite en géométrie algébrique. Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 4. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. ix+133 pp. DM 23.60.

The aim of this book is to "commencer à donner un exposé aussi complet que possible des fondements de la Géométrie Algébrique abstraite." Its contents can be described roughly as most of Weil's "Foundations of algebraic geometry" [Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; MR 9, 303], together with certain other foundational topics such as Chow coordinates and specializations of cycles, some of Zariski's work on local theory over arbitrary ground fields, and the author's own partial extension in his thesis [J. Math. Pures Appl. (9) 30 (1951), 159-205, 207-274; MR 13, 980] of intersection theory to components of intersection that may be of excess dimension or singular on the ambient variety. The theory is developed systematically and in such a way as to include the most general versions of the fundamental facts; for example, irreducibility, normality, and simplicity are discussed first with respect to a fixed ground field, the corresponding "absolute" concepts being introduced later by studies of the behaviour of the relative properties upon ground-field extensions. The volume is kept to a manageable size by omitting material that might be called "purely algebraic" (merely giving the necessary definitions, statements, and references in a brief "rappel algébrique" at the end) and by condensation of the proofs. The book is not intended to be an easy introduction to the field (though, as a matter of fact, large parts can be read

by anyone with a reasonable knowledge of commutative algebra, and suitable amplification could make most of the rest comprehensible to such a person); rather, it is designed to be of help to algebraic geometers. For these, its utility will consist in the systematic presentation of the fundamental facts in their most general contexts, together with many simplified proofs of more esoteric results.

The book consists of two parts, the first of which deals with the elementary global theory of affine and projective varieties. For this part one needs little more than a good general knowledge of field theory and specializations, and the "rappel algébrique" system works quite well. Topics covered are algebraic sets in affine and projective space, irreducibility, the Hilbert Nullstellensatz, dimension, products and projections, the dimension of intersections, normalization, ground-field extensions, Chow coordinates, specializations of cycles, and algebraic correspondences. The second part deals with local theory and intersection multiplicities. Here the basic reference is to the author's "Algèbre locale" [Gauthier-Villars, Paris, 1953; MR 14, 1012]. Topics covered include the local ring of a point and of a subvariety, normal points and the analytic irreducibility and analytic normality of varieties at normal points, simple points (including Zariski's jacobian criterion for simplicity relative to a fixed ground field), and intersection theory. Using the author's definition of the multiplicity of an ideal primary for the maximal ideal of a local ring, the basic properties of intersection multiplicities for varieties in affine and projective space are derived in an efficient 15 pages, after which follow such items as the multiplicity of subvarieties, divisors of functions, relative intersection theory, and the theorem of Severi to the effect that any cycle in a projective space is a complete intersection of divisors. The part on multiplicities is extremely technical and mere reference to the "rappel algébrique" is not sufficient to follow the details.

The given version of the "projection formula" (p. 81) is not correct; the hypothesis relative to the variety at infinity must be strengthened. A number of other minor errors, mostly typographical, are easy to catch. The book closes with a brief historical note, a brief bibliography, a dictionary comparing the terminologies of various authors, and an index. As a comment on progress we remark that a similar terminological dictionary in Weil's "Foundations" is a 2 by 7 table; here it is 7 by 15. M. Rosenlicht.

See also: Perron, p. 229.

NUMERICAL ANALYSIS

Frank, Evelyn. On the calculation of the roots of equations. J. Math. Phys. 34 (1955), 187-197.

L'auteur propose une méthode de calcul approché des racines d'un système d'équations. La base est la formule de Newton poussée jusqu'aux ordres supérieurs (c'est-à-dire un développement suivant les puissances des valeurs des fonctions). Le calcul est présenté sous une forme commode à utiliser numériquement. Deux exemples sont traités.

J. Kuntzmann (Grenoble).

Benson, G. C., and Schreiber, H. P. A method for the evaluation of some lattice sums occurring in calculations of physical properties of crystals. II. Canad. J. Phys. 33 (1955), 529-540.

In a previous paper by van der Hoff and Benson [same J. 31 (1953), 1087-1094; MR 15, 352] a method was

described for evaluating sums of the type

$$\sum' (k^2 + l^2 + m^2)^{-n}$$

taken over the infinite cubic lattice. The present authors consider, more generally, sums of the type

$$T_n(r, s) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} k^r l^s (k^2 + l^2 + m^2)^{-n}$$

where r, s, n are positive integers for which $2n > r + s + 3$ and discuss the application of the methods of the previous paper. This requires the evaluation of the double sum

$$S_v(r, s) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k^r l^s (k^2 + l^2)^{-v} = S_v(s, r),$$

where $v = n - \frac{1}{2}$. By applying the reduction formula

$$S_v(r, s) = S_{v-1}(r-2, s) - S_v(r-2, s+2),$$

the case in which r, s are not both odd can be reduced to evaluating $S_r(0, q)$ by previous methods. For r, s both odd, the problem reduces to $S_r(1, q)$ which cannot be evaluated by previous formulas. The sum $S_r(1, 1)$ having important crystallographic application, a forthright evaluation by the FERUT computer was undertaken. The authors tabulate $S_r(1, 1)$ for $2v=5(1)30$ to 8 significant digits. The first of these values $S_{5/2}(1, 1)=0.43636718$ required 10000 terms of the series.

D. H. Lehmer (Berkeley, Calif.).

Macon, Nathaniel. On the computation of exponential and hyperbolic functions using continued fractions. J. Assoc. Comput. Mach. 2 (1955), 262-266.

Here is described the rapid computation of e^x , $\tanh x$, $\sinh x$ and $\cosh x$ by means of the even parts of known continued-fraction expansions for these functions. A bound for the approximants is given. E. Frank.

Ostrowski, A. M. Note on a logarithm algorithm. Math. Tables Aids Comput 9 (1955), 65-68.

The author comments further on the logarithm algorithm discussed by D. Shanks [same journal 8 (1954), 60-64; MR 15, 830]. If $a_0 > a_1 > 1$, and the sequences of numbers a_2, a_3, \dots and positive integers n_1, n_2, \dots are determined successively by the relations

$$a_i^{n_i} \leq a_{i-1} < a_i^{n_i+1}, \quad a_{i+1} = a_{i-1}/a_i^{n_i} \quad (i=1, 2, \dots),$$

then $\lambda = \log a_0 a_1$ is given by the continued fraction

$$\lambda = \frac{1}{n_1 + \frac{1}{n_2 + \dots}}$$

If the i th convergent of λ is $\mu_i = P_i/Q_i$, then the author shows that, if $\mu = \ln a_0$ and γ_0 is the positive root of the equation $e^x = 1 + \mu + \mu^2$ (so that $\gamma_0 \approx 1.7933$), then, as soon as $\mu/Q_i \gamma_0$, $\lambda - \mu_i = (-1)^i (a_{i+1} - 1)/\mu Q_i$ with an error bounded by μ/Q_i^3 . Thus, by computing

$$[P_i + (-1)^i (a_{i+1} - 1)\mu^{-1}]/Q_i$$

instead of μ_i , an increase in accuracy is obtained which is equivalent to the saving of about onethird of the steps of the original algorithm. S. Gorn.

Arend, S. Interpolation et extrapolation basées sur la méthode des moindres carrés. L'exploitation des données empiriques. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 52 (1955), pp. 1-8. 850 francs. The Cracovian product of two matrices A and B , taken in that order, is the ordinary matrix product $B^T A$. A preamble of two pages describes the Cracovian algebra, and the paper itself sets forth in this algebra elementary formulas for interpolation and least squares.

A. S. Householder (Oak Ridge, Tenn.).

Dommanget, J. Détermination des paramètres de la fonction $y=a+b/(x, \alpha, \beta, \dots)$ sur la base du principe des moindres carrés. Application au cas de la fonction exponentielle. L'exploitation des données empiriques. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 52 (1955), pp. 9-28. 850 francs.

When the function f depends upon only the single parameter α , then upon differentiating the sum of squares one obtains three linear equations in a and b whose coefficients are functions of α . Hence, one equates the determinant to zero and solves for α , and the a and b are thus readily obtained. The author now observes that

this choice of α maximizes a certain function, and that, in fact, when there are additional parameters β, γ, \dots , the function is maximized with respect to all the variables. The special case $y=a+b \exp(\alpha x)$ is treated in some detail.

A. S. Householder (Oak Ridge, Tenn.).

Gale, L. A. A modified-equations method for the least-squares solution of condition equations. Trans. Amer. Geophys. Union 36 (1955), 779-791.

The least-squares problem here differs slightly from the usual one. A vector V of minimal length is required which satisfies the condition $AV=W$, where A has r rows and $s > r$ columns. Hence, $V=A^T K$ where $AA^T K=W$. The main problem is, therefore, solving for K . The method recommended for this is due to Bjerhammar [Trans. Roy. Inst. Tech. Stockholm no. 49 (1951); MR 14, 1127] which amounts to performing a Schmidt orthogonalization of the rows of A and yields a factorization $AA^T = \Delta \phi \Delta^T$, where Δ is unit upper triangular and ϕ diagonal (the author says, "orthogonal"). An advantage is, of course, that when additional equations are adjoined the matrices Δ and ϕ are enlarged but not otherwise modified.

A. S. Householder (Oak Ridge, Tenn.).

Cvetkov, B. A new method of computation in the theory of least squares. Austral. J. Appl. Sci. 6 (1955), 274-280.

In matrix notation, the quadratic form $x^T P x$, where P is diagonal, is to be minimized subject to the side conditions $Ax=u$. If k is the vector of Lagrange multipliers, then $Px=A^T k$, and $x=P^{-1}A^T k$ can be substituted into the side conditions for the determination of k . The problem arises in geodetic work.

The author recommends a Schmidt orthogonalization of the side conditions with respect to P . That is, one forms $A=VB$, where V is lower triangular and $BP^{-1}B^T$ is diagonal. Advantages: Unsuspected dependencies are often exhibited; new conditions can be adjoined without requiring any repetition of the work. Moreover, it is remarked that in practice some of the equations are already orthogonal at the start. Two numerical examples are given. Orthogonalization in simple regression has, of course, been recommended many times, but the case with side conditions seems to be less common.

A. S. Householder (Oak Ridge, Tenn.).

Gani, J., and Moran, P. A. P. The solution of dam equations by Monte Carlo methods. Austral. J. Appl. Sci. 6 (1955), 267-273.

Given a stochastic matrix, it is well known that the largest latent root is $\lambda=1$; that under rather general conditions the root is simple; and that the associated vector has only non-negative elements. This represents the steady state, and Monte Carlo is an obvious possibility for estimating the elements of the vector. The authors consider a matrix of order 7 representing the release of water from dams and compare the results after 100, 500, and 1000 trials and those obtained by direct solution, estimating standard errors also by splitting the total series into subseries. They estimate that 529,000 trials would be required to reduce the standard deviation to 10^{-3} . It is suggested that whereas the method is not to be recommended for the particular matrix treated, it might be practical for larger matrices giving a finer representation of a process that is in fact continuous.

A. S. Householder (Oak Ridge, Tenn.).

★Hochstrasser, Urs. Die Anwendung der Methode der konjugierten Gradienten und ihrer Modifikationen auf die Lösung linearer Randwertprobleme. Dissertation, Eidgenössische Technische Hochschule in Zürich, 1954. i+46 pp.

Let (*) $Dx + l = 0$ be a finite system of linear algebraic equations with a positive-definite symmetric matrix D . The author first reviews the method of conjugate gradients for solving (*) [M. R. Hestenes and E. Stiefel, J. Res. Nat. Bur. Standards 49 (1953), 409-436; MR 13, 651]. To reduce the labor in the applications the author considers leaving constant the two scalar parameters which Hestenes and Stiefel recompute at each step. He is thus led to the "second-order Richardson" procedure of S. P. Frankel [Math. Tables Aids Comput. 4 (1950), 65-75; MR 13, 692]. Letting $r_k = Dx_k + l$, one has $r_k = (1 + \epsilon)r_{k-1} - \lambda Dr_{k-1} - \epsilon r_{k-2}$. The author discusses the method and its convergence for different ϵ , λ ; when $\epsilon = 0$ one has the "Gesamtschritt" (ordinary Richardson) method.

When D corresponds to the difference equations for an elliptic boundary-value problem, the author interprets the various algorithms as methods of solving initial-value problems for the time-dependent problem $\partial x / \partial t + Dx + l = 0$. The paper has several practical observations on such matters as computational stability, storage, initial guesses, etc. In a calculation on an IBM card-programmed calculator the Frankel and Gesamtschritt methods were compared for the Poisson difference equation over a rectangle.

In the last two sections the author uses a variational approach and a triangular net to set up difference equations in the two coordinates of the displacement vector for a certain plane stress-strain problem. The 106 equations in 106 unknowns were satisfactorily solved in 90 steps of the conjugate gradient method on the Zuse computer in Zurich.

G. E. Forsythe (Los Angeles, Calif.).

Fraeys de Veubeke, B. M. Iteration in semidefinite eigenvalue problems. J. Aero. Sci. 22 (1955), 710-720.

The author is dealing with the eigenvalue problem associated with the equation $(C - \omega^2 M)x = 0$ where C is a symmetrical semidefinite positive matrix of stiffness coefficients, M is a symmetrical positive definite matrix of inertia coefficients, ω is circular frequency, and x the vector of vibration amplitudes. Since C is semidefinite there are zero eigenvalues with associated free modes u_i such that $Cu_i = 0$ ($i = 1, 2, \dots, m$). For example, in case of an airplane, the free modes might correspond to the rigid motions such as pitching, rolling, and yawing. The elastic modes x_i are those associated with non-zero eigenvalues, and hence such that $(C - \omega_i^2 M)x_i = 0$. The paper is devoted to a discussion of various methods for finding the lower non-zero eigenvalues and their modes. A simple example is worked out numerically in some detail to illustrate some of the methods discussed. W. E. Milne.

Dingle, R. B. The evaluation of integrals containing a parameter. Appl. Sci. Res. B. 4 (1955), 401-410.

The author discusses the following methods for the computation of integrals involving a parameter, for small or large values of that parameter: expansion of the integrand, integrations by parts, Mellin transformation (with respect to the parameter), differential equations (with the parameter as the independent variable of the differential equation). The discussion is illustrated by application of

the various methods to

$$\int_0^\infty u^{-n} e^{-xu} du.$$

A. Erdélyi (Pasadena, Calif.).

Dingle, R. B. The integrals $C_n(x) = \int_1^\infty u^{-n} \cos ux du$ and $Si_n(x) = \int_1^\infty u^{-n} \sin ux du$ and their tabulation. Appl. Sci. Res. B. 4 (1955), 411-420.

The properties and applications of the integrals of the title are discussed, and the methods of the paper reviewed above are applied to obtain several expansions for these functions. [The author does not seem to be aware of the closely related results of Böhmer, Differenzengleichungen, Koehler, Leipzig, 1939; Bateman, Proc. Nat. Acad. Sci. U.S.A. 32 (1946), 70-72; MR 7, 442; Kreyszig, Acta Math. 85 (1951), 117-181; 89 (1953), 107-131; MR 12, 825; 14, 871.] 6D tables of $C_n(x)$ and $Si_n(x)$ are given for $n = -1(1)6$, $x = 0(1)1(2)5(5)20$. A. Erdélyi.

Dingle, R. B. Tables of the integrals

$$CI_n(x) = \int_1^\infty u^{-n} (1-u^2) \cos ux du$$

and

$$SI_n(x) = \int_1^\infty u^{-n} (1-u^2) \sin ux du.$$

Appl. Sci. Res. B. 4 (1955), 421-424.

In terms of the integrals defined in the preceding review, $CI_n(x) = C_{n+2}(x)$, $SI_n(x) = Si_{n+2}(x)$. Here 6D tables of $CI_n(x)$ and $SI_n(x)$ are given for $n = -2(1)4$, $x = 0(1)1(2)5(5)20$. A. Erdélyi.

Fettis, Henry E. Numerical calculation of certain definite integrals by Poisson's summation formula. Math. Tables Aids Comput. 9 (1955), 85-92.

L'auteur déduit de la formule

$$\int_0^\infty f(x) dx = h \left[\frac{1}{2} f(0) + \sum_{k=1}^\infty f(kh) \right] - 2 \sum_{k=1}^\infty g(2k\pi/h),$$

$$g(x) = \int_0^\infty f(t) \cos(xt) dt$$

des expressions approchées de diverses fonctions, en particulier, $J_0(x)$ ($0 \leq x \leq 2$) à 10^{-3} près par une formule à 4 termes, $J_0(x)$ ($0 \leq x \leq 11$) à 10^{-10} près. L'auteur donne encore d'autres formules relatives à $I_0(x)$, à des fonctions de Bessel modifiées de 2ème espèce, à la fonction erreur.

J. Kuntzmann (Grenoble).

Hammer, Preston C., and Hollingsworth, Jack W. Trapezoidal methods of approximating solutions of differential equations. Math. Tables Aids Comput. 9 (1955), 92-96.

Les auteurs considèrent dans l'intégration des équations différentielles non seulement les valeurs discrètes mais les fonctions simples qui approchent la solution dans les divers intervalles. Par exemple, la formule de Milne $y_1 - y_0 = (y_0' + y_1')h/2$ conduit à $u(x) = y_0 + (x - x_0)y_0' + (x - x_0)^2(y_1' - y_0')/2h$. On obtient des formules nouvelles en imposant à un polynôme du 2ème degré de satisfaire à l'équation différentielle en deux points de l'intervalle. On peut prendre en particulier les racines du polynôme de Legendre de degré deux. La possibilité d'utiliser effectivement ces formules est discutée. J. Kuntzmann.

Stein, P. A numerical solution of $\frac{d^2y}{dx^2} = F(x)$. Math. Gaz. 39 (1955), 203-206.

To solve the boundary problem $\frac{d^2y}{dx^2} = F(x)$ with conditions $y(0) = y_0$, $y(l) = y_1$, the author divides the interval $(0, l)$ into $n+1$ equal parts at the points x_1, x_2, \dots, x_n . He replaces the differential equation by a difference equation which is rigorously equivalent to the differential equation, $y(x_{i+1}) - 2y(x_i) + y(x_{i-1})) = -\lambda(x_i)$, and shows that this can be solved explicitly in simple form for the $y(x_i)$ in terms of the quantities $\lambda(x_i)$. These latter depend on $F(x)$ and on the values y_0 and y_1 . The case where the n th derivative of $F(x)$ has finite discontinuities is also treated.

W. E. Milne.

Lyusternik, L. A. On eigenvalues of finite-difference approximations of the Laplace operator. Dokl. Akad. Nauk SSSR (N.S.) 89 (1953), 613-616. (Russian)

Finite-difference approximations to differential operators possess certain "parasitic" properties, belonging to the particular difference operator rather than to the original differential operator being approximated. The author has already studied [Trudy Mat. Inst. Steklov. 20 (1947); MR 10, 71] from this point of view the "simplest" difference approximation to the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In the present note such a study is carried out for a much more extended class of difference operators which approximate the same Laplace operator. [For a detailed presentation of the announced results see the paper reviewed second below.]

J. B. Diaz.

Lyusternik, L. A. On general network approximations of the Laplace operator. Dokl. Akad. Nauk SSSR (N.S.) 91 (1953), 1267-1269. (Russian)

This note is a continuation of the paper reviewed above which was concerned with the eigenvalues and eigenfunctions of certain general difference operators which approximate the Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In the present note the questions of the convergence of the eigenvalues and eigenfunctions of the difference operators to the corresponding quantities for the differential operator are scrutinized. [For a detailed account of the results see the paper reviewed below.]

J. B. Diaz.

Lyusternik, L. A. On difference approximations of the Laplace operator. Uspehi Mat. Nauk (N.S.) 9 (1954), no. 2(60), 3-66. (Russian)

The present paper is concerned with questions relative to certain difference operators which "approximate" the Laplace differential operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Let A, B, \dots designate points (vectors) of the plane, and suppose $c(A, B) = d(A, B-A)$ is a real-valued function, satisfying the following conditions: (1) $c(A, B) \geq 0$; (2) given A , one has $c(A, B) \neq 0$ only for a finite number of points B ; (3) $c(A, B) = c(B, A)$; (4) $\sum_B c(A, B) = 1$. Then a typical ("averaging") operator acting on a function $f(A)$ is given by:

$$Sf(A) = \sum_B c(A, B)f(B) = \sum_i c(A, A+A_i)f(A+A_i) = \sum_i d(A, A_i)f(A+A_i),$$

upon writing $d(A, A_i) = c(A, A+A_i)$. (Given A , the points B for which $c(A, B) \neq 0$ are called the "neighbors" of A and are denoted by A_i .) Of special interest are the "homogeneous" operators, for which $c(A, B) = d(A, B-A)$

depends only on $B-A$. For these homogeneous S one has

$$Sf(A) = \sum_i d(A_i)f(A+A_i) = \sum_i d_i f(A+A_i) \quad (d(A_i) = d_i),$$

where: (1) $d(A_i) \geq 0$, (2) $d(A_i) \neq 0$ only for a finite number of points A_i , (3) $d(A_i) = d(-A_i)$, (4) $\sum_i d(A_i) = 1$. Further, putting $A_i = (x_i, y_i)$, and defining S_A by $S_A f(A) = \sum_i d_i f(A+A_i)$, one has that

$$S_A f(A) = f(A) + Mh^2 \Delta f(A) + o(h^2),$$

where $\sum_i x_i^2 = \sum_i y_i^2 = 2M$. These operators occur in the approximation of Laplace's differential operator by difference operators [e.g., S. E. Mikeladze, Izv. Akad. Nauk SSSR. Ser. Mat. 1938, 271-292; L. Collatz, Numerische Behandlung von Differentialgleichungen, Springer, Berlin, 1951; MR 13, 285]. The paper contains a detailed study of the eigenvalues and eigenfunctions of these operators and their "convergence" or "divergence" to the corresponding eigenvalues and eigenfunctions of the Laplace differential operator. The mixed initial-boundary-value problems for the heat equation $\partial u/\partial t = \Delta u$ and the wave equation $\partial^2 u/\partial t^2 = \Delta u$ are also considered from this difference-approximation point of view [convergence and divergence in these cases are very closely connected with the "stability" and "instability" in the sense of O'Brien, Hyman, and Kaplan J. Math. Phys. 29 (1951), 223-251; MR 12, 751]. The derivation of "probable" (as opposed to "exact") estimates of the error is also studied.

J. B. Diaz (College Park, Md.).

Hara, I. S. On a method of approximate conformal mapping of a many cornered region onto the unit circle. Dopovidi Akad. Nauk Ukrain. RSR 1953, 289-293. (Ukrainian. Russian summary)

A numerical method for approximating the constants which occur in the application of the Christoffel-Schwartz method for the conformal representation of polygonal regions is given with examples.

C. Saltzer.

Gardner, J. W., Gellman, H., and Messel, H. Numerical calculations on the fluctuation problem in cascade theory. Nuovo Cimento (10) 2 (1955), 58-74.

Bolin, Bert. Numerical forecasting with the barotropic model. Tellus 7 (1955), 27-49.

Charney, J. The use of the primitive equations of motion in numerical prediction. Tellus 7 (1955), 22-26.

Hovanskii, G. S. A method of construction of nomograms with oriented transparencies. Vyčisl. Mat. Vyčisl. Tehn. 2 (1955), 3-93. (Russian)

This paper provides practical methods for constructing nomograms with a movable plane that can be attached to a drafting machine or used with a T-square. This plane carries only scales (graduated or ungraduated) or fixed points. The systems of equations representable in this way

$$\begin{aligned} f_{12} - f_7 &= f_{34} - f_8 = f_{56} - f_9, \\ g_{12} - g_7 &= g_{34} - g_8 = g_{56} - g_9, \end{aligned}$$

are specialized in various ways to obtain the most adaptable standard forms. A mnemonic is developed and in a given case it is applied to first write the simplest form of the equations for the scales and binary fields to produce a schematic. The equations are then expanded to contain transformation parameters. Twenty-nine tables of equa-

tions are given for the forms considered. Optimal choice of the parameters and their geometric significance receive attention. Comparisons are made with some of the canonical forms for alignment charts. Five applications are presented in detail. *R. Church* (Monterey, Calif.).

Bal, Lascu, et Rado, Francisc. Deux théorèmes relatifs à la séparation des variables d'une équation à cinq variables. *Com. Acad. R. P. Romine* 5 (1955), 285-290. (Romanian. Russian and French summaries)
Necessary and sufficient conditions are given for the equation $F(x, y, z, u, v) = 0$ to be solvable for v in the form $v = \Phi[\phi(x, y, z), \psi(z, u)]$ or the form $v = \Phi[\phi(x, y, z), u]$. The idea is to apply these results to the construction of nomograms. *D. H. Lehmer* (Berkeley, Calif.).

Bal, Lascu, et Rado, Francisc. La séparation des variables en nomographie. *Com. Acad. R. P. Romine* 5 (1955), 303-305. (Romanian. Russian and French summaries)
The results of the paper reviewed above are generalized to the case of functions F of $n+1$ variables. *D. H. Lehmer* (Berkeley, Calif.).

Bal, Lascu, et Rusu, Ioan. Sur un groupement de variables en vue de la construction des nomogrammes composés. *Acad. R. P. Romine. Fil. Cluj. Stud. Cerc. Şti.* 5 (1954), 45-49. (Romanian. Russian and French summaries)
The results of the paper reviewed above are generalized to the case in which the equation $F=0$ is equivalent to a system of p equations $\lambda_i = \phi$ where ϕ is a function of λ_{i-1} and some of the original variables of F . *D. H. Lehmer*.

Cseke, V., et Csendes, Z. Quelques problèmes pratiques concernant la construction de n nomogrammes pour les équations du type $f_3(w) = f_1(u) \cdot f_2(v)$. *Acad. R. P. Romine. Fil. Cluj. Stud. Cerc. Şti.* 5 (1954), 51-58 (1 plate). (Romanian. Russian and French summaries)
The authors discuss the practical difficulties in constructing nomograms for equations of the type indicated in the title. An example from probability theory is worked out in considerable detail. *D. H. Lehmer*.

Bonfiglioli, Luisa. Nomograph for two common-axis scales and one oblique scale for functions of three variables. *Riveon Lematematika* 9 (1955), 29-35. (Hebrew. English summary)

See also: Harrik, p. 256; Slobodyanskii, p. 266, 286; Daugaret, p. 268; Jordan, p. 277; Birkhoff, Goldstine, and Zarantonello, p. 309; Goldstine and von Neumann, p. 314.

Mathematical Machines

Gutenmacher, L. I., Kuz'minok, G. K., and Klabukova, L. S. A method of solving linear algebraic equations on a vacuum-tube integrator. *Vychisl. Mat. Vychisl. Tehn.* 2 (1955), 230-246 (2 plates). (Russian)
(The reviewer is using his own notation.) Let the rectangular real matrix A of rank m have n rows and m columns ($m \leq n$). The author wants to find x minimizing $|Ax - c|^2$, using a vacuum-tube integrator. It is observed that all eigenvalues of the partitioned matrix

$$\mathfrak{A} = \begin{bmatrix} 0 & -A^T \\ A & I \end{bmatrix} \quad (T \text{ denotes transpose})$$

have positive real part, thus insuring stability of the following machine method: Solve the system

$$b \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \mathfrak{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

on the integrator. Then $\lim_{t \rightarrow \infty} x(t)$ will minimize $|Ax - c|^2$. If $m=n$ one is solving $Ax=c$ stably without the Gaussian pretransformation to $A^T A x = A^T c$. Examples of orders 2 to 6 are discussed, with photographs of the oscilloscope. *G. E. Forsythe* (New York, N.Y.).

Boscher, Jean. Sur la détermination analogique de la fonction d'Airy dans des domaines multiplement connexes. *C. R. Acad. Sci. Paris* 241 (1955), 1023-1025.

Mohanti, H. B., and Booth, A. D. A simple electronic Fourier synthesizer. *J. Sci. Instrum.* 32 (1955), 442-444.

See also: Tasny-Tschiasny, p. 328.

RELATIVITY

Urbah, V. Yu. Generalized theory of vector fields. *Dokl. Akad. Nauk SSSR (N.S.)* 101 (1955), 1043-1046. (Russian)

A classical generalization of the Maxwell theory is constructed by assuming a non-Riemannian space with metric $ds = \gamma_i dx^i$. Tensor suffixes are raised and lowered by means of the auxiliary tensor

$$b_{ik} = \frac{1}{2}(\gamma_i \gamma_k + \gamma_k \gamma_i) e_i, \quad e_1 = e_2 = e_3 = 1, \quad e_4 = -1.$$

The equation of a geodesic takes the usual form

$$(\frac{d^2 x^i}{ds^2}) + \Gamma_{ik}^i (\frac{dx^k}{ds}) (\frac{dx^i}{ds}) = 0$$

with the affine connection given by

$$\Gamma_{ik,l} = \frac{1}{2}(\Gamma_{il} \gamma_k + \Gamma_{kl} \gamma_i), \\ \Gamma_{km} = (\partial \gamma_m / \partial x^k) - (\partial \gamma_k / \partial x^m).$$

The curvature tensor $B_{ikl}{}^m$ and the Einstein tensor R_{ik} are defined in the usual way in terms of the $\Gamma_{ik,l}$. Field

equations are derived from the action principle $\delta S = 0$ with $S = \int R_{ik} b^{ik} dx^4$. After eliminating surface terms, the action integral reduces to the form

$$S = \frac{1}{2} \int (\Gamma_{im} \Gamma^{im}) b^4 dx^4.$$

Thus the linear approximation to this theory is identical with the Maxwell theory which has the Lagrangian

$$L = -(16\pi)^{-1} F_{im} F^{im}.$$

An exact solution is obtained for the static spherically symmetric case. The scalar potential ϕ is then

$$\phi = \varepsilon \lambda^{-1} [\exp(\lambda/r) - 1],$$

where ε and λ are constants of integration, having dimensions of charge and length respectively. This gives a modification of the Coulomb law by a very strong repulsive force within the particle radius λ .

The theory is also worked out for a vector meson field,

obtained from the above formalism by adding a supplementary term quadratic in the γ_k to the Lagrangian.
F. J. Dyson (Princeton, N.J.).

Ingraham, R. L. An extension of the Maxwell theory. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 165-169.

The extension of the Maxwell theory referred to in the title consists of replacing the six components of the antisymmetric tensor in Minkowski space by the ten components F_{ab} of an antisymmetric tensor in a five-dimensional Riemannian space whose points are the spheres of the Minkowski space and whose metric is the infinitesimal angle between such spheres. The components of the extended tensor are to be functions of five variables and to satisfy the conditions: (1) F_{ab} is a curl; (2) the divergence of F_{ab} vanishes. The equations which are to play the role of the Lorentz ponderomotive-force equations are taken to be the analogues of the corresponding equations in special relativity.
A. H. Taub (Urbana, Ill.).

Mavridès, Stamatia. Choix de la métrique et du champ électromagnétique en théorie unitaire d'Einstein. Lien avec la théorie de Born-Infeld. J. Phys. Radium (8) 16 (1955), 482-488.

The author gives a brief discussion of some of the difficulties in choosing suitable definitions of the metric and electromagnetic fields in Einstein's Unified Theory. Several definitions are examined in detail. It is shown that it is possible to choose definitions such that the electromagnetic field remains finite at the origin and that integral of the density of charge over all space remains finite and represents the charge of a particle.
M. Wyman (Edmonton, Alta.).

Mavridès, Stamatia. La solution générale des équations d'Einstein $g^{\mu\nu}_{; \mu} = 0$. C. R. Acad. Sci. Paris 241 (1955), 173-174.

Many authors, including Tonnelat, Hlavatý and others have discussed the general solution of Einstein equations $g^{\mu\nu}_{; \mu} = 0$. The present author uses similar methods to discuss the equations $g^{\mu\nu}_{; \mu} = 0$.
M. Wyman.

Ikeda, Mineo. On static solutions of Einstein's generalized theory of gravitation. II. Progr. Theoret. Phys. 13 (1955), 265-275.

[For part I see same journal 12 (1954), 7-30; MR 16, 531.] The author examines some of the known solutions of Einstein's Unified Field Theory and shows that these solutions have singularities which cannot be transformed away by purely radial transformations. He agrees with Einstein that it is a singularity-free solution that is required to represent the physical fields. From this point of view he goes on to show that a single magnetic pole cannot exist. He concludes that this is a feature which is favorable to the new theory.
M. Wyman (Edmonton, Alta.).

Scherrer, Willy. Zur linearen Feldtheorie. III. Die Gravitationsgleichungen. Z. Physik 140 (1955), 374-385.

The author restricts his theory propounded in previous papers [Z. Physik 139 (1954), 16-34, 44-55; 140 (1955), 160-163, 164-180; MR 16, 79, 635, 756] to deal with the so called symmetric case (pure gravitation). He then computes $R_{ij} - \frac{1}{2}g_{ij}R$ and compares it to the corresponding quantity in the general theory of relativity; here R_{ij} is the Ricci tensor of the Riemannian space with the metric tensor g_{ij} and R is the scalar curvature of this space.
A. H. Taub (Urbana, Ill.).

Murai, Yasuhisa. Wave equations in conformal space. Wave equation for nucleon. Sci. Rep. Saitama Univ. Ser. A. 2 (1955), 7-31.

The author considers the five-dimensional space of spheres in a Minkowski space in terms of six homogeneous coordinates. In this representation the conformal group of the Minkowski space is the projective group of a quadric. The formalism of the author differs from and is related to that of Ingraham in the paper reviewed below. This paper is mainly concerned with the properties of various tensor and spinor wave equations in the five-dimensional space of spheres and their [provisional] interpretation in terms of the Minkowski space.
A. H. Taub (Urbana, Ill.).

Ingraham, R. L. Conformal geometry and elementary particles. Nuovo Cimento (9) 12 (1954), 825-851.

The author discusses the correspondence between the spheres in R_4 (a flat space of signature $(+ + + -)$), and the points of a five-dimensional space of constant unit curvature whose length element is the infinitesimal angle between two neighboring spheres. "A test particle at an event" is represented by a point in the latter space, that is by a 5-uple of numbers (x, y, z, t, w) . The first four are the coordinates of the center of the sphere in R_4 and the fifth coordinate is a length associated with the "mass" of the "particle" in R_4 by the relation $m = \hbar w/c$, \hbar being Planck's constant divided by 2π . The motion of free test particles is described by the geodesics of the five-dimensional space. The null geodesics are said to be the paths of free electrons or free protons and the non-null ones paths of free elementary particles.
A. H. Taub.

Ingraham, R. L. The behavior of finite particles in conformal-covariant weak field theory. Nuovo Cimento (10) 1 (1955), 82-102.

The field equations and the equations of motion of test particles of the conformal relativity theory previously propounded by the author [see the paper reviewed above] are linearised. Solutions of these approximate equations are obtained and discussed. The connection between radiation damping and some non-Maxwellian terms of the equations of motion is examined. It is claimed that every exact solution of the theory of general relativity is properly contained in this conformal relativity in the sense that every exact solution of the former gives a "pure gravitational" exact solution of the latter. The reviewer cannot agree with this statement, for the theory of a perfect electrically neutral fluid moving in its own gravitational field does not seem to be contained in the conformal relativity theory discussed here.
A. H. Taub.

Just, Kurt. Zur Kosmologie mit veränderlicher Gravitationszahl. Z. Physik 140 (1955), 648-655.

The author studies the field equations for a cosmological space in which the gravitational constant varies in accordance with the Jordan theory [Schwerkraft und Weltall, Vieweg, Braunschweig; 1952; MR 14, 1022]. For those models in which there is a linear expansion in time, various constants must be restricted in range. These restrictions differ from those imposed on the Jordan theory by other considerations.
A. H. Taub.

Sen, N. R., and Roy, T. C. On a steady gravitational field of a star cluster free from singularities. Z. Astrophys. 34 (1954), 84-90.

The authors obtain a static spherically symmetric

solution of the relativistic field equations and discuss the applicability of their solution to a star cluster.

M. Wyman (Edmonton, Alta.).

Park, David. Radiations from a spinning rod. *Phys. Rev.* (2) **99** (1955), 1324-1325.

Starting from the linearized energy-momentum pseudotensor, the author calculates the radiation pattern for the gravitational radiation from a slowly-spinning narrow rod. He compares it with the pattern for a corresponding distribution of charge. *F. A. E. Pirani* (London).

Payne, W. T. Spinor theory and relativity. I. *Amer. J. Phys.* **23** (1955), 526-536.

Amar, Henri. New geometric representation of the Lorentz transformation. *Amer. J. Phys.* **23** (1955), 487-489.

Furry, W. H. Lorentz transformation and the Thomas precession. *Amer. J. Phys.* **23** (1955), 517-525.

See also: Kilmister, p. 298; Panofsky and Phillips, p. 326.

MECHANICS

Pöschl, Th. Bemerkung über Stoßprobleme für verbundene Systeme nach der Lagrangeschen Methode. *Österreich. Ing.-Arch.* **9** (1955), 216-217.

Starting from the linearized energy-momentum pseudotensor, the author calculates the radiation pattern for the gravitational radiation from a slowly-spinning narrow rod. He compares it with the pattern for a corresponding distribution of charge. *F. A. E. Pirani* (London).

O. Bottema (Delft).

Cattaneo, Carlo. Sulla „sufficienza“ del principio dei lavori virtuali all'equilibrio di un generico sistema materiale. *Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl.* (5) **14** (1954), 209-220.

A rigorous derivation of the sufficiency of the principle of virtual work. At the foundation are hypotheses of regarding the constraints and the forces, assumed to depend on position and velocity. Also involved is a hypothesis on the passivity of reaction and an elementary theorem on kinetic energy. The proof consists in showing that under the given conditions, the kinetic energy is necessarily zero. *D. C. Lewis.*

Salvadori, Luigi. Sulla stabilità dei moti merostatici di particolari sistemi anolonomi. *Rend. Accad. Sci. Fis. Mat. Napoli* (4) **20** (1953), 66-78 (1954).

There is defined a rather restricted class of conservative non-holonomic dynamical systems for which it is possible to formulate criteria of stability and instability of merostatic motion analogous to the classical criterion of Dirichlet for stability of equilibrium in a holonomic conservative system. The method of proof is quite similar to that used in the paper reviewed below. *D. C. Lewis.*

Salvadori, Luigi. Un'osservazione su di un criterio di stabilità del Routh. *Rend. Accad. Sci. Fis. Mat. Napoli* (4) **20** (1953), 269-272 (1954).

Let q_1, \dots, q_m be ignorable coordinates in a conservative holonomic dynamical system with n ($> m$) degrees of freedom. Let T be the kinetic energy and $-U$ the potential energy. Then the first integrals $\partial T / \partial q_i = c_i$ ($i=1, \dots, m$) may be used to eliminate the first m q 's from the kinetic energy in such a way that $T = T_2 + T_0$, where T_2 is a quadratic form in $\dot{q}_{m+1}, \dots, \dot{q}_{m+n}$, while T_0 is a function of q_{m+1}, \dots, q_{m+n} and the c 's, but does not depend on any of the q 's. It is then shown that any point in $(2n-m)$ -dimensional space $(\dot{q}_1, \dots, \dot{q}_n, q_{m+1}, \dots, q_{m+n})$ where $T_0 - U$ has a minimum considered as a function of q_{m+1}, \dots, q_{m+n} alone is stable with respect to all sufficiently small perturbations and not merely those satisfying the m first integrals of ignorance. *D. C. Lewis.*

★ **García, Godofredo.** Absolute form of the transformation of the equations of dynamics in a curved space of n dimensions. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 87-111. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

The author treats the following question: Given two dynamical systems, under what conditions is it possible to set up a correspondence between the values of the coordinates and time variables of the two systems such that trajectories are mapped into trajectories in a one-to-one way? The question has been treated previously by others, notably by T. Y. Thomas [*J. Math. Phys.* **25** (1946), 191-208; *MR* **8**, 102]. The present paper consists largely of mere successions of equations, with no accompanying discussion; and no theorems or other summary statements of the results are given. Consequently, it is difficult to determine precisely what the results are. Some of the author's remarks indicate that they are essentially the same as the results obtained by Thomas.

L. A. MacColl (New York, N.Y.).

de Castro Brzezicki, A. An equation concerning central motion. *Rev. Mat. Hisp.-Amer.* (4) **15** (1955), 3-8. (Spanish)

A particle moves in a plane subject to a force which is central and depends on the position and velocity of the particle. The author seeks conditions on the force which are necessary and sufficient in order that the orbit of the particle shall belong to an arbitrarily prescribed two-parameter family of curves. The conditions are found in the form of a certain partial differential equation. The general solution of the equation is given, and some particular cases are discussed.

L. A. MacColl (New York, N.Y.).

Colombo, G. Sulle orbite periodiche di un sistema conservativo in due gradi di libertà. *Univ. e Politec. Torino. Rend. Sem. Mat.* **13** (1954), 327-333.

Certain fairly general types of Hamiltonian systems with two degrees of freedom, having a given periodic solution, are characterized in such a way that the existence of at least one further periodic solution with the same energy can be proved from consideration of a surface of section, topologically a 2-cell, in connection with the Brouwer fixed-point theorem. The periodic solution thus obtained has the property that one of the coordinates assumes each value between its minimum and maximum just twice in each period. *D. C. Lewis.*

Mihailović, Dobrovoje. *Beitrag zur Untersuchung eines besonderen Falles der Bewegung im Widerstandsmittel*. Bull. Soc. Math. Phys. Serbie 6 (1954), 102-107. (Serbo-Croatian. German summary)

G. N. Dubošin [Astr. Ž. 9 (1932), 20-26] considered a particular case of the motion of a particle in a resisting medium. The author treats the same problem as a perturbed Keplerian motion and gives a geometrical interpretation of the Dubošin quasi-integral.

E. Leimanis (Vancouver, B.C.).

Sokolov, Yu. D. *On approximate solution of the basic equation of the dynamics of a hoisting cable*. Dopovidi Akad. Nauk Ukrain. RSR 1955, 21-25. (Ukrainian. Russian summary)

The partial differential equation obtained by G. N. Savin [Dokl. Akad. Nauk SSSR (N.S.) 97 (1954), 991-994; Ukrain. Mat. Ž. 6 (1954), 126-139; MR 16, 533, 1060] for a shaft hoisting cable is considered under the assumption that the lifting of the attached load (resting on a fixed support) proceeds according to a trapezoidal tachogram with constant acceleration. An approximate solution is given by averaging the inertia terms in the above mentioned equation over the interval $0 \leq x \leq l$, where l is the natural length of the cable. For the "zero phase" of the lifting process (while the load rests on the support) the problem is reduced to the integration of an ordinary second-order linear differential equation. For the remaining three phases (as distinguished by the author) a system of two ordinary second-order linear differential equations is obtained. The dynamic forces applied to the lower end of the cable are determined by an approximate ordinary fourth-order linear differential equation.

E. Leimanis (Vancouver, B.C.).

Egerváry, Jenő. *Matrix theory applied to the calculation of suspension bridges*. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 303-313. (Hungarian)
Hungarian version of Acta Tech. Acad. Sci. Hungar. 11 (1955), 241-256; MR 16, 894].

See also: Butenin, p. 265; Coburn, p. 297; Lighthill and Whitham, p. 310.

Fluid Mechanics, Acoustics

*Kotschin, N. J., Kibel, I. A., und Rose, N. W. *Theoretische Hydromechanik. Band II*. Akademie-Verlag, Berlin, 1955. viii+569 pp. DM 48.00.

This is a translation by K. Krienes of a Russian textbook [Kočin, Kibel' i Roze, *Teoreticheskaya gidrodinamika*, Čast' 2, 3rd ed., Gostehizdat, Moscow, 1948]. According to an inserted notice, the volume is intended as a textbook for universities and technical institutes in East Germany. There are three major divisions of subject matter: theoretical gas-dynamics, viscous-fluid flow, and the elements of turbulence theory; these have been written by Kibel, Kotschin, and Kibel, respectively (Rose's contributions are in the first volume).

The section devoted to gas-dynamics treats all of the classical subject matter of this field, including the theory of characteristics, shock waves, hodograph techniques, and linear approximations. The methods of Hristianović, based on the hodograph equations, are given in detail and form the basis for both the familiar

Kármán-Tsien approximation and an approximate procedure for supersonic flow. The treatment, throughout, seems more mathematical than physical; e.g., shock waves are always described simply as discontinuity surfaces, discussion of their real structure being postponed to the second section.

The section on viscous fluids is mostly classical in its content and approach, which begins with the stress tensor and proceeds through exact solutions, Stokes's and Oseen's approximations, to boundary-layer theory. However, the Oseen theory of fluids with vanishing viscosity, which has received little attention in recent years, is also included at some length. For the compressible boundary layer the Dorodnicyn method is given; the Stewartson-illingworth transformation was not known in 1948. The Becker theory of shock-wave structure is given.

The final division of the book, devoted to turbulence, is the shortest of the three (64 pages). It begins with a rather complete exposition of the theory of hydrodynamic stability, including the Tollmien-Schlichting-Lin theory for boundary layers. Modern developments pertaining to isotropic and locally-isotropic turbulence are not presented in any detail but are only mentioned. Mixing-length and similarity theories are presented, though rather briefly.

Throughout the volume there are footnotes indicating that small corrections have been made by the translator. There is also a brief addendum to the German edition and seven new references have been added. W. R. Sears.

*Iacob, Caius. *Introducere matematică în mecanica fluidelor*. [Mathematical introduction to the mechanics of fluids.] Editura Academiei Republicii Populare Române, Bucurest, 1952. 838 pp. Lei 24.00.

This book is a well organized, coherent, elaborately documented account of lectures given at the University of Cluj in 1947-1949 and material presented at conferences in Bucharest in 1950. It provides excellent coverage of certain topics in classical hydrodynamics and compressible-flow theory up to 1950. It deals almost entirely with steady irrotational non-viscous flows. Except for chapters on determination of velocity fields corresponding to given vorticity fields, on Prandtl's lifting-line theory of wings of finite span, and sections on conical supersonic flow fields, the problems considered are strictly two-dimensional. As the title implies, the point of view is exclusively theoretical; there is no extended consideration of calculation of aerodynamic coefficients, practical treatment of wind-tunnel corrections, etc. The author's more than twenty years' research on boundary-value problems for plane harmonic functions is reflected in a thorough account of the potential-theoretical background for the solution of plane incompressible-flow and linearized compressible-flow problems. The section on the theory of hyperbolic partial differential equations is not up to the same standard, and there is no discussion of methods of characteristics. Since there is no mention of approximate theories of transonic flow, presumably the book was written before they had become widely known.

Part I (174 pp.) deals with Green's functions and conformal representation; the Dirichlet, Neumann, and other problems for the circle, circular annulus, etc.; and the solution of these problems with the aid of the theory of integral equations. Part II (176 pp.) discusses the equations of motion, computation of forces on an obstacle, determination of incompressible velocity fields from

vorticity fields, and the fundamental ideas of two-dimensional incompressible flow. Part III (116 pp.) on theories of resistance of incompressible fluids discusses in considerable detail the theory of Helmholtz separated flows with stagnant wakes, flows in two-dimensional jets, and various generalizations. It also contains short chapters on plane airfoil theory and the theory of lifting wings of finite span. Part IV (184 pp.) on compressible fluids is devoted to the propagation of waves of discontinuity, generalities on the theory of second-order partial differential equations, and extended discussion of applications of Chaplygin's hodograph method, as well as such standard topics of supersonic flow as simple waves, shock waves, and axisymmetric flows about circular cones. Part V (176 pp.) discusses various methods of approximate solution of the problems of fluid mechanics, including of course the Janzen-Rayleigh method and linearization. There is also a section on general linearized supersonic conical flows. *J. H. Giese (Aberdeen, Md.).*

Iacob, Caius. Sur le calcul de la pression qu'exerce un courant liquide, variable avec la hauteur, sur un obstacle mobile. Acad. R. P. Romine. Bul. Şti. Secţ. Şti. Mat. Fiz. 6 (1954), 801-809. (Romanian. Russian and French summaries)

The author extends the Blasius formulae for force and moment to a profile moving with linear and angular acceleration in a stream which at infinity has the velocity $V(\theta) + 2\alpha y$ parallel to the x -axis. This generalizes the formulae of Y. H. Kuo [Quart. Appl. Math. 1 (1943), 273-275; MR 5, 80] for steady shear flow disturbed by a profile. *L. M. Milne-Thomson (Greenwich).*

Panda, J. N. A note on the circle theorem in hydrodynamics. Amer. Math. Monthly 62 (1955), 576.

Proof of a theorem of Milne-Thomson [Proc. Cambridge Philos. Soc. 36 (1940), 246-247; MR 1, 284].

Prem Prakash. On a flow superposable on the two-dimensional radial flow. Ganita 5 (1954), 21-24.

Prem Prakash. Self-superposability in axially symmetric flows. Proc. Nat. Inst. Sci. India. Part A. 21 (1955), 1-7.

The author continues his study [Ganita 2 (1951), 75-80; 3 (1952), 91-93; Bull. Calcutta Math. Soc. 45 (1953), 51-54; Math. Student 22 (1954), 129-135; MR 15, 476; 14, 1029; 15, 905; 16, 759] of superposable flow in the sense of Ballabh [Proc. Benares Math. Soc. (N.S.) 2 (1940), 69-79, 85-89; J. Indian Math. Soc. (N.S.) 16, 191-197 (1953); MR 3, 283; 14, 1029]. In this paper conditions are developed for an axially-symmetric flow to be self-superposable, and these are then used to determine special solutions of the Navier-Stokes equations. By this means the author obtains several new unsteady flows for which the Stokes streamfunction is a function of the radial coordinate (and time). *J. B. Serrin.*

Barua, S. N. A source in a rotating fluid. Quart. J. Mech. Appl. Math. 8 (1955), 22-29.

The ultimate steady flow due to a point source placed on the axis of an initially uniformly rotating liquid of infinite extent is studied. It is assumed that the irrotational flow due to the source is confined to the region inside a cylindrical discontinuity surface which extends to infinity along the axis of rotation and has a bulge in the neighborhood of the source. A number of additional

rather severe assumptions are introduced to determine the flow and the shape of the discontinuity surface.

G. W. Morgan (Providence, R.I.).

Rahnberg, Gösta. On the rectilinear vortex filament in a cylinder. Appl. Sci. Research A. 5 (1954), 12-30.

This paper considers several examples of plane potential flows that contain a vortex in their interior and are bounded by constant-speed free streamlines and polygonal walls. The author is primarily concerned with the induced motion of the vortex. *D. Gilbarg.*

Dom, U. Ein Beitrag zur Stabilitätstheorie der Wirbelstrassen unter Berücksichtigung endlicher und zeitlich wachsender Wirbelkerndurchmesser. Ing.-Arch. 22 (1954), 400-410.

This is an effort to bring the theory of the vortex street in line with observation by taking account of diffusion of vorticity. The author proposes a model of the vortex street in which the vortices decay like independent stationary vortices in a viscous fluid, the velocity at any point in this approximation being assumed equal to the sum of the induced velocities of the separate vortices, all of which start simultaneously as classical potentials. The author carries out an analysis of stability under small disturbances much as in the classical Kármán theory, treating the case of periodic disturbances in a single row, and of a single perturbed vortex in the alternating and symmetrical double rows. The mathematical details differ from those in the classical theory in that the linearized equations for small disturbances contain explicit time-dependent terms. As expected, the single row is always unstable, but unlike the Kármán theory, the alternating double row has, in a certain sense, a domain of stability described in terms of a time-dependent width-spacing ratio. There is also a stability domain for the symmetrical double row, but this can be reached only by passing through a regime of instability. The results coincide with those of the Kármán theory for small values of viscosity and time. *D. Gilbarg (Bloomington, Ind.).*

Dom, Ulrich. The stability of vortex streets with consideration of the spread of vorticity of the individual vortices. J. Aero. Sci. 22 (1955), 750-754.

This investigation concerns the stability of vortex streets consisting of two parallel rows of staggered real vortices of identical structure whose vorticity spreads with time into the ambient fluid. It is found that for the case of stability the ratio of center-line width h to pitch l of the street is a function of the non-dimensional time $\tau = (4\nu/l^2)t$, where ν is the kinematic viscosity and t the time. For vanishing viscosity and also for vanishing time, the value calculated by von Kármán, $h/l = 0.281$, is obtained. (Author's summary.) *D. W. Dunn.*

★ **Zarantonello, E. H.** Hydrodynamics: recent advances in the theory of free boundaries. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América. Julio, 1954, pp. 113-128. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

This paper contains an excellent review of recent (up to 1954) progress in the theory of the Helmholtz-Kirchhoff free-boundary problem. After stating the problem, the author turns to a discussion of the associated integral equations and to the existence theorems (both for com-

pressible and incompressible fluids) obtained from the integral equations by means of the Schauder-Leray fixed-point theorem. He next considers the variational methods which have been exploited in the treatment of axially symmetric problems. There follows a discussion of the uniqueness problem and the important advances made possible here by the comparison method, and, in particular, Lavrentieff's theorem. The two final sections are devoted to the constructive and numerical methods for obtaining the solution which the author has helped to develop. The paper contains a 31-item bibliography, limited to papers of theoretical value which introduce new ideas and methods. *J. B. Serrin* (Minneapolis, Minn.).

Birkhoff, G., Goldstine, H. H., and Zarantonello, E. H. Calculation of plane cavity flows past curved obstacles. Univ. e Politec. Torino. Rend. Sem. Mat. 13 (1954), 205-224.

This paper describes the first systematic calculations ever carried out on cavity and jet flows past curved obstacles, comprising altogether fifty symmetric plane flows past convex and concave bodies, under several different conditions of streamline detachment. The computations were performed on a high speed digital computer, using an improved version of a scheme previously described by Birkhoff, Young, and Zarantonello [Proc. Symposia Appl. Math. 4 (1953), 117-140; MR 15, 258], which is an extension of that used by Brodetsky [Proc. Roy. Soc. London. Ser. A 102 (1923), 542-553] in his calculation of Helmholtz flows past circular and elliptic cylinders. The basic idea is to solve approximately the functional equation associated with the free-boundary problem by fitting the curvature of the body at N points corresponding to equally spaced values of the independent variable in the functional equation, and solving the resulting finite system of equations by a suitable iterative scheme. Although this procedure has been proved, by Zarantonello [Collect. Math. 5 (1952), 175-225; MR 15, 571], to converge to the true solution as $N \rightarrow \infty$ only in the special case that the parameter describing the nature of the free streamline detachment is a prescribed constant, the actual calculations exhibit convergence in all cases that have been tried. The flow problems discussed here were coded for $N=24$, corresponding to 12 points on each side of the axis of symmetry (as contrasted with 3 used by Brodetsky). The authors discuss in detail the problems of numerical analysis arising from the calculations. In particular, they find the computed curvature to be usually accurate at interpolated points to better than 1 percent, and often to .2 percent, except near the point of detachment, where the errors are in the range 3-10 percent for flows in which the free streamline has infinite curvature at separation. The authors conclude from their analysis that the functional equation used by them is not well suited to accurate work when the curvature of the body varies by a large factor of say 10 or more, and that equal spacing of the mesh points will not in general give accurate results at the point of detachment unless $N \gg 24$.

D. Gilbarg (Bloomington, Ind.).

Hahnemann, H. W. Konturen von freien Ausflußstrahlen und ihre technischen Anwendungen. Forsch. Gebiete Ingenieurwesens 18 (1952), 45-55.

The author considers the efflux of a jet from an orifice. As is well known, in the case of two-dimensional flows the hodograph method and function theory provide a complete solution if the containing vessel is polygonal and

gravity is neglected. The paper treats in this manner several cases of engineering interest and then carries over to axially symmetric flows the values of the contraction coefficient derived from the exact two-dimensional theory. This procedure is based on experimental results mentioned by the author and on the well known computations of Trefftz [Z. Math. Phys. 64 (1916), 34-61] which imply equality between the contraction coefficients of the plane and axially symmetric vena contracta. [Recent calculations by Garabedian [Appl. Math. Statist. Lab., Stanford Univ., Tech. Rept. no. 42 (1955)] yield a contraction coefficient of .58 for the axially symmetric vena contracta as compared with .61 in the plane case, and cast doubt on the validity of Trefftz' results.] *D. Gilbarg*.

Broflos, Ath. Les forces de choc d'un corps solide sur un fluide réel. Cas de la sphère. Génie Civil 132(1955), 128-130.

Essentially an expository discussion of some known methods for calculating the force on a body moving through a fluid or during surface impact. *D. Gilbarg*.

Tan, H. S. On source and vortex of fluctuating strength travelling beneath a free surface. Quart. Appl. Math. 13 (1955), 314-317.

The motion is assumed two-dimensional, the depth of fluid is infinite, the singularity travels with constant horizontal velocity c , its strength varies as $\exp(i\omega t)$, and the equations of motion are linearized. By use of analytic continuation the velocity potential is found very simply in the form of an integral. The author observes that a single travelling vortex cannot occur alone. [The reviewer notes that the condition at infinity ahead of the singularity is incomplete. When $4\omega c < g$, it should be replaced by a radiation condition.] *F. Ursell*.

Lighthill, M. J., and Whitham, G. B. On kinematic waves. I. Flood movement in long rivers. Proc. Roy. Soc. London. Ser. A. 229 (1955), 281-316.

The paper contains the theory of a type of wave motion which arises in any one-dimensional flow problem when there exists an approximate functional relation at each point between the rate of flow q of a quantity and the concentration k of this quantity. The wave property then follows from the equation of continuity satisfied by q and k , the wave velocity being $\partial q / \partial k$. These "kinematic" waves may coalesce to form "kinematic shock waves". The shock waves are in reality narrow regions in which terms neglected in the q - k relation become important.

The general theory is applied to a detailed treatment of flood movement in rivers where the q - k relation (k is the volume of water per unit length) arises essentially by assuming a balance between the force of turbulent friction on the river bed and the component of gravity parallel to the free surface, the fluid inertia being neglected. A discussion of the various factors influencing the q - k relation is given. The role of ordinary long gravity waves in flood movement is then investigated and it is shown that under conditions appropriate to flood waves, the gravity waves are rapidly attenuated and the main disturbance is carried downstream by the more slowly travelling kinematic waves. The application of kinematic wave theory to the determination of flood movement is presented. The effect of tributaries and run-off is considered, and finally modifications of the theory to include slight dependence of the q - k relation on derivatives of q or k are discussed.

G. W. Morgan.

Lighthill, M. J., and Whitham, G. B. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.* 229 (1955), 317-345.

The uni-directional flow of traffic on a crowded divided highway is treated as a problem in kinematics of continua. A functional relationship is stipulated between the flow q (number of vehicles passing a stationary observer per unit of time) and the concentration k (number of vehicles per unit of length of the highway). The "conservation of vehicles" is then expressed by a quasi-linear partial differential equation of the first order for the concentration. For a homogeneous stretch of road, the characteristics in the (t, x) -plane are straight lines of the slope dq/dk and concentration and flow are constant along any characteristic. A given "state" (k, q) propagates with the (group) velocity dq/dk , whereas a given vehicle advances with the (phase) velocity q/k , which, for the assumed relation between k and q , exceeds the group velocity. In regions of the (t, x) -plane where characteristics intersect, discontinuities of flow and concentration must be considered. If the jumps in these quantities across a discontinuity are denoted by Δq and Δk , the discontinuity is found to propagate with the speed $\Delta q/\Delta k$. The consideration of non-homogeneous road conditions leads to a theory of bottlenecks. Queuing at a traffic light is treated as an example. Various secondary effects are briefly discussed.

W. Prager (Providence, R.I.).

★ Robinson, A. On some problems of unsteady aerofoil theory. *Proceedings of the Second Canadian Symposium on Aerodynamics*, Toronto, 1954, pp. 106-122. The Institute of Aerophysics University of Toronto, Toronto, 1954.

The pressure distribution on a two-dimensional airfoil that flies with variable velocity along a curved flight path in an incompressible flow and executes a prescribed, time-dependent, transverse deformation is determined on the basis of linearized theory; the result is a generalization of Küssner's solution for constant flight velocity [*Luftfahrtforschung* 13 (1936), 410-424]. It is shown that the problem of flight with variable speed along a quasi-circular path may be reduced to an equivalent problem of flight with constant speed. The paper concludes with a discussion and criticism of the usual concept of an aerodynamic derivative.

J. W. Miles.

Synge, J. L. The motion of a viscous fluid conducting heat. *Quart. Appl. Math.* 13 (1955), 271-278.

L'auteur étudie les mouvements tridimensionnels d'un fluide compressible visqueux et conducteur de chaleur; il estime commode de faire figurer parmi les fonctions inconnues l'entropie spécifique et le volume spécifique. Les équations du mouvement sont établies dans le cas le plus général et sont ensuite "linéarisées" au voisinage d'un état d'équilibre. Les solutions exponentielles élémentaires représentent des perturbations qui se propagent dans le fluide; elles dépendent de quatre constantes, en général complexes. L'auteur distingue suivant la valeur des constantes trois types de perturbations; certaines sont planes, d'autres sont stationnaires.

H. Cabannes.

Pieruschka, E. Ergänzung zu dem Aufsatz "Die mathematischen Grundlagen zu einer Messmethode des Schubmoduls zäher Flüssigkeiten". *Z. Angew. Math. Mech.* 34 (1954), 192-193.

In a previous work [same *Z.* 31 (1951), 83-92; MR 16,

190] the author applied results concerning plane shear flow to experiments in which the parallel plates are replaced by coaxial cylinders. Now he gives the correction terms resulting when the same method of linearization is applied to the equations in cylindrical coordinates.

C. A. Truesdell (Bloomington, Ind.).

Kapica, P. L. The hydrodynamic theory of lubrication with rolling. *Z. Tehn. Fiz.* 25 (1955), 747-762. (Russian)

The problem is that of the lubrication of the cylinders or balls in a bearing. The main, but by no means all, the assumptions are that inertia and body force in the oil may be neglected and that the pressure and viscosity are related by the law $\mu = \mu_0 \exp(\gamma p)$, where μ_0 and γ are constants. On this basis the author concludes that the layer of oil between cylinder and bearing has a minimum thickness which further pressure will not reduce, and that the area of oil in contact with the rolling cylinder is greater than in the static case. From this he infers that rolling considerably decreases the stresses in the metal.

L. M. Milne-Thomson (Greenwich).

Olzak, Wacław, and Litwiniszyn, Jerzy. Sur un phénomène non-linéaire d'écoulement d'un liquide comme un modèle rhéologique. *Arch. Mech. Stos.* 5 (1953), 557-583 (1954). (Polish. Russian and French summaries)

A hydraulic model is proposed for the representation of non-linear creep and relaxation, consisting of a fluid container of suitably specified form with an orifice at the bottom or communicating with a cylindrical container in which a piston is moving. The differential equation for the fluid level in the container with open orifice or for the level of the piston in the communicating containers as a function of time neglecting inertia forces is considered as a creep equation the non-linearity of which can be arbitrarily specified by the shape of the container. The advantage of the proposed model over a directly formulated non-linear creep-equation is not apparent.

A. M. Freudenthal (New York, N.Y.).

Anzelius, Adolf. Two flow problems in a viscous fluid. *Ark. Fys.* 9 (1955), 391-398.

The author considers the plane flow of a viscous incompressible fluid in a sector or a strip. Approximate solutions of the Navier-Stokes equations are obtained by expanding the stream function in inverse powers of the kinematic viscosity and using standard perturbation techniques. The higher-order perturbations quickly become complicated, but calculations are carried out for six special cases; the corresponding streamline patterns are illustrated.

J. B. Serrin (Minneapolis, Minn.).

Stuart, J. T. On the effects of uniform suction on the steady flow due to a rotating disk. *Quart. J. Mech. Appl. Math.* 7 (1954), 446-457.

From the author's summary: "The exact ordinary differential equations of von Kármán for the flow due to a rotating disk of infinite radius are integrated for the case of uniform suction through the disk. In the analysis a suction parameter a is introduced, where $a(v_0)$ is the velocity of suction, ν being the kinematic viscosity and ω the angular velocity of the disk. For $a=1$ the equations are integrated numerically, but for higher values of a a series solution in descending powers a is obtained." The

author discusses in detail the dependence of the various flow quantities on the suction parameter. *D. Gilbarg.*

Chandrasekhar, S., and Elbert, Donna D. The instability of a layer of fluid heated below and subject to Coriolis forces. II. *Proc. Roy. Soc. London. Ser. A.* 231 (1955), 198-210.

This paper is an extension of an earlier paper by Chandrasekhar [same *Proc.* 217 (1953), 306-321; MR 15, 174]. In the first paper it was shown that the effect of the Coriolis force was to inhibit the onset of convection. It was also found that if the Coriolis force is considered it was possible that thermal instability could set in as oscillations of increasing amplitude (over-stability). Which type of instability will occur depends upon the ratio of the kinematic viscosity, ν , and the thermometric conductivity, k . The present paper is concerned with determining the critical Rayleigh number for over-stability. This is done by use of a variational method. The cases considered are (a) both boundary surfaces free (also considered in the first paper), (b) both boundary surfaces rigid, and (c) one boundary surface free, the other rigid. Numerical results are presented for the case $\nu/k = .025$ (equal to that for mercury at ordinary temperatures). For this value of ν/k over-stability can occur. The theoretical results for case (b) are compared with the experimental results obtained by Fultz and Nakagawa [*Proc. Roy. Soc. London. Ser. A.* 231 (1955), 211-225]. The agreement is satisfactory. *R. C. DiPrima (Cambridge, Mass.).*

Wadhwa, Y. D. On boundary layer thickness. *Z. Angew. Math. Mech.* 35 (1955), 295-300. (German, French and Russian summaries)

The author shows that under the action of special body forces, the boundary-layer thickness over a parabolic cylinder is of the order $Re^{-1/2}$ where Re is the Reynolds number and λ is a positive number less than $\frac{1}{2}$. The solution chosen is made to satisfy the Navier-Stokes equations by adjusting a system of body forces. For this reason, the solution is unrealistic. *Y. H. Kuo (Ithaca, N.Y.).*

Gregory, N., Stuart, J. T., and Walker, W. S. On the stability of three-dimensional boundary layers with application to the flow due to a rotating disk. *Philos. Trans. Roy. Soc. London. Ser. A.* 248 (1955), 155-199.

This paper is divided into three parts. The first part relates the experimental work performed by Gregory and Walker. They first discuss briefly experiments on the stability of viscous flow over swept wings; but in the main they are concerned with the flow due to a rotating disk. Velocity profiles for laminar and turbulent flow are determined; the laminar flow profile is in good agreement with the theoretical results; however the agreement of experiment and theory for turbulent flow is not as satisfactory. In the stability analysis it is found that amplified waves arise which lead rapidly to a transition of the flow. The waves cover a certain band of frequencies (measured at a point fixed in space) and some of the waves are found to be stationary relative to the surface of the disk. These waves give rise to an observed vortex pattern, whose axes take the form of equi-angular spirals, bounded by radii of instability and transition. These radii are tabulated for various values of the angular velocity of the disk.

The second part of the paper is a report on the theoretical analysis by J. Stuart. The general equations of motion in orthogonal curvilinear co-ordinates are ex-

amined by superimposing an infinitesimal disturbance periodic in space and time on the main flow, and linearizing for small disturbances. It is found in formulating the eigenvalue that within the range of certain approximations the problem is reduced to the two-dimensional Orr-Sommerfeld equation where the mean flow velocity is taken in the direction in which the disturbance propagates. Hence to study instability a whole class of velocity profiles must be considered corresponding to the different directions of propagation of the disturbance. It is to be emphasized that these results are applicable only to possible modes of disturbance in a local region, otherwise flow curvature must be considered. The question of instability at infinite Reynolds number is examined in detail. Tollmien's criterion [*Nachr. Ges. Wiss. Göttingen. Fachgruppe I. (N.F.)* 1 (1935), 79-114] that velocity profiles with a point of inflection are unstable at infinite Reynolds number is extended to the case of profiles with two critical points. Corresponding to the experimentally observed vortices, one velocity profile is found which generates neutral disturbances of zero phase velocity. A variational method is derived and used to compute the wave number. The fixed vortices predicted by the theory have as their axis equiangular spirals of angle 103° in good agreement with experiment, but the agreement between theoretical and experimental wave number is not good, the discrepancy being attributed to viscosity. Finally there is a discussion of the problem by Gregory and Stuart. *R. C. DiPrima (Cambridge, Mass.).*

Dunn, D. W., and Lin, C. C. On the stability of the laminar boundary layer in a compressible fluid. *J. Aero. Sci.* 22 (1955), 455-477.

In this paper a more complete theory for the stability of the laminar boundary layer in a compressible fluid than that presented by Lees and Lin [*NACA Tech. Note no. 1115* (1946); MR 8, 236] is developed. The main changes are the inclusion of three-dimensional disturbances and the demonstration that, in many cases, the stability characteristics depend on temperature fluctuations (contrary to previous conclusions).

The linearized equations for small three-disturbances in a flow of the boundary-layer type are reduced to a simplified system by a careful order-of-magnitude analysis of the complete equations. The error in the simplified equations for moderate Mach numbers is $O(R^{-1/2})$ where R is the Reynolds number based on boundary-layer thickness. For very high Mach numbers the simplified equations remain the same but the error estimate is not as good.

Solutions of the disturbance equations of the form $Q'(x, y, z, t) = q(y) \exp \{i(\alpha x + \beta z - \alpha ct)\}$ are sought; the imaginary part of c of course determining stability or instability. It is possible by an appropriate transformation of variables to reduce the eighth-order system for three-dimensional disturbances to a sixth-order two-dimensional system. Although this new system does not represent a proper two-dimensional disturbance, still it does make the mathematical treatment considerably simpler. Following the theory of Lees and Lin, the secular equation for determining the Reynolds number for neutral stability (i.e. $c_r = 0$) is formulated. It is found that the secular equation contains a term which depends on the inviscid and viscous solutions as well as the thermal boundary conditions. This term does not appear in the treatment by Lees and Lin. In forming this secular equation, improved solutions of the disturbance equations are used. The rather complex and detailed mathematical

discussions of these solutions are to be presented in a future paper; however the results are briefly summarized in the present paper.

Some of the conclusions reached by the authors are the following: For subsonic Mach numbers three-dimensional disturbances are less important than two-dimensional disturbances, except possibly under conditions of extreme cooling. As the Mach number increases three-dimensional disturbances become more important, and for free-stream Mach numbers between one and two they begin to play the leading role in many problems of practical interest. At supersonic free stream Mach numbers the boundary layer can never be completely stabilized with respect to all three-dimensional disturbances by cooling of the surface; however it is shown that cooling is effective in stabilizing the boundary layer for Mach numbers up to two—this conclusion would probably remain unchanged for Mach numbers up to six, say. *R. C. DiPrima.*

Tannenwald, L. M. Asymptotic spherical shock decay. *J. Appl. Phys.* 26 (1955), 551-555.

The author's concern is with the decay of a weak spherical shock, taking account of viscosity and thermal conductivity. To reduce the mathematical theory to tractable proportions he makes several physical assumptions which imply that decaying and steady-state shock fronts of equal strength behave similarly in the neighborhood of the shock front and that the spherical shock front may be considered plane to a high approximation. These assumptions, which are combined with known results on the isentropic theory of shock decay by means of an interpolation scheme, lead finally to an ordinary first-order differential equation relating the strength of the shock and its radius. The author discusses briefly a program of numerical work designed to obtain a solution in actual practise. *D. Gilbarg* (Bloomington, Ind.).

Litwiniszyn, Jerzy. On a certain problem of the two-dimensional turbulent flow. *Arch. Mech. Stos.* 5 (1953), 273-290. (Polish. English summary)

L'auteur commence par une discussion sur quelques théories fondamentales de turbulence. Après cette exposition l'auteur développe sa théorie propre, qui est basée sur la notion de métré-holonomie [voir Schouten, *Tensor analysis for physicists*, Oxford, 1951; MR 13, 493]. En particulier l'auteur montre que, en certains cas, le phénomène du mouvement de la chaleur dans un médiateur en mouvement peut être considéré comme un temps-espace modèle de variation des composants de vitesse secondaire de turbulence. *K. Bhagwandin* (Oslo).

Shigemitsu, Yutaka. Statistical theory of turbulence. I. Hypothesis of vortex chaos motion. *J. Phys. Soc. Japan* 10 (1955), 472-482.

L'auteur se propose de construire une théorie de la turbulence reposant sur l'irrégularité chaotique de l'agitation turbulente. De l'examen de divers aspects expérimentaux des phénomènes turbulents, il tente de déduire une formulation mathématique. Se limitant d'abord au cas de deux dimensions, il définit une densité de probabilité $P(r, \theta)$ par un procédé qui semble le suivant: A étant un point du fluide, on considère les centres des tourbillons qui défilent sur A pendant un temps T . Ils remplissent un domaine D . Certains d'entre eux traversent un élément d'aire dS de D , dont le centre a pour coordonnées polaires r, θ . Soit $T_{r, \theta}$ la somme des intervalles de temps correspondants. $P(r, \theta)dS$ est égal à $\lim_{T \rightarrow \infty} T_{r, \theta}/T$.

Cette définition, qui joue le rôle d'une hypothèse ergodique, permet de transformer les moyennes temporelles en moyennes sur le domaine D , et de donner des expressions des grandeurs utilisées d'habitude dans la théorie de la turbulence. Ces expressions peuvent être reportées dans les équations de Reynolds. Le travail s'achève par quelques considérations sur la forme que peut revêtir la fonction $P(r, \theta)$ suivant le degré d'irrégularité du mouvement tourbillonnaire. *J. Bass* (Paris).

Shigemitsu, Yutaka. Statistical theory of turbulence. II. Foundation of similarity theory of turbulence. *J. Phys. Soc. Japan* 10 (1955), 890-902.

L'auteur applique les méthodes qu'il a introduites dans la première partie (analysé ci-dessus) de son étude à la discussion des lois de similitude dans trois types d'écoulements turbulents: (A) Turbulence sans tension: la distribution de la vitesse d'agitation est alors normale sans corrélations, et il y a isotropie lorsque la vitesse moyenne est uniforme. (B) Turbulence décroissante avec tensions (cas d'un sillage). (C) Turbulence non décroissante avec tensions (cas de la couche limite turbulente). *J. Bass.*

Codegone, Cesare. Sulle definizioni di gas perfetto. *Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* 89 (1955), 93-96.

The author points out that various definitions of "perfect gas" in terms of thermostatic parameters differ from one another. He feels that because of this the usual construction of thermostatics on the basis of a Carnot cycle for perfect gases may be unsound.

C. A. Truesdell (Bloomington, Ind.).

Tao, L. N. Gas dynamic behavior of real gases. *J. Aero. Sci.* 22 (1955), 763-744, 794.

L'auteur étudie le comportement d'un gaz dont l'équation d'état est la suivante

$$p = RT\varrho^2(1 - c\varrho T^{-3})[\varrho^{-1} + B_0(1 - b\varrho)] - A_0(1 - a\varrho)\varrho^2;$$

p, T, ϱ désignent la pression, la température absolue et la masse spécifique; R, a, b, c, A_0 et B_0 sont des constantes. Un tel fluide est comparé à un fluide idéal qui obéirait à l'équation d'état $p = RT\varrho$. Les résultats concernent les écoulements isentropiques, les chocs normaux et l'angle de Mach dans l'écoulement après un choc oblique.

H. Cabannes (Marseille).

Seth, B. R. New formulation of equations of compressible flow. *Bull. Calcutta Math. Soc.* 46 (1954), 217-220.

By manipulation of the equation satisfied by the velocity potential defining an isentropic compressible flow, the author rediscovers the Prandtl-Meyer flow around a corner. By violating the equation of state, an exact solution for three-dimensional case is also found.

Y. H. Kuo (Ithaca, N.Y.).

Schubert, Hans, und Schincke, Erich. Zur Ermittlung von Unterschallströmungen mit der Transformationsmethode bei quadratischer Approximation der Adiabate. *Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl.* 101 (1955), no. 6, 32 pp.

Consider a steady plane potential flow with pressure p , density ϱ , speed of sound a , polar coordinates w, θ in the hodograph plane, and stream function Ψ . Let $\ln \omega = \int_{\infty}^w (1 - w^2/a^2)^{1/2} w^{-1} dw$ and $K = (1 - w^2/a^2)^{1/2} \varrho_0/a$, where subscripts 0 will generally denote conditions at infinity. If the

pressure density relation is such that

$$(*) \quad K = (1 - w_0^2/a_0^2)^{1/2} \{J_0(2^{-1/2}k\omega)/J_0(2^{-1/2}k)\}^2,$$

then $\psi(\omega, \theta) = K^{1/2} \Psi(w, \theta)$ satisfies

$$(**) \quad \Delta \psi + \frac{1}{2} k^2 \psi = 0.$$

The author sets $a^2/a_0^2 = 1 + \sum_{n=1}^{\infty} a_n(q-q_0)^n$ and $\varrho/\varrho_0 = 1 + \sum_{n=1}^{\infty} b_n(q-q_0)^n$, where $q = w^2/a_0^2$, and determines ϕ from $d\phi/dq = a^2$. Bernoulli's equation and (*) determine all a_n and b_n except a_1 . Finally, $a_1, k, \phi_0, \varrho_0$, and a_0 can be chosen so that $\phi/\phi_0 - (\varrho/\varrho_0)^{1/2}$ is of third order in $q - q_0$. The author then solves (**) by separating variables and expands $\psi = \psi_1 + \psi_2$ in series of products of Bessel and trigonometric functions. ψ_2 is regular near the image $\theta = \omega = O(w_0, 0)$ of the velocity at infinity. A particular singular solution ψ_1 is chosen so that Ψ will have the same type of singularity at $\theta = \omega = 0$ as the stream function for the incompressible flow about a circle. An example which retains only the first term of the series for ψ_2 is considered at length and is found by numerical integration to yield a subsonic flow about an approximately circular obstacle.

J. Giese (Aberdeen, Md.).

Jordan, Peter F. Some series developments in unsteady aerodynamics. *J. Aero. Sci.* 22 (1955), 722-724.

Dans cette courte note, l'auteur propose de nombreux développements en série, en général assez simples, pour remplacer certaines formules, classiques en aérodynamique non stationnaire, ordinairement écrites avec des intégrales. Les expressions proposées peuvent être avantageuses dans certains calculs numériques.

P. Germain.

Matunobu, Yaso'o. Application of the thin-wing-expansion method to the compressible flow past a Kaplan bump. *J. Phys. Soc. Japan* 10 (1955), 814-822.

The paper is concerned with the determination of the velocity distribution over a cusped aerofoil, which has come to be called the Kaplan bump. The velocity distribution over the aerofoil is found, correct to $O(t^2)$ (where t is the thickness ratio), using the thin-wing-expansion method due to Imai [Rep. Aero. Res. Inst. Tokyo Imp. Univ. no. 294 (1944)]. This problem has been solved previously by Kaplan [NACA Rep. no. 768 (1943); MR 9, 477] to the same order in the thickness parameter; however, the method used in the present paper is considerably simpler since it is only necessary to calculate certain quantities on the aerofoil. Kaplan's method requires a knowledge of these quantities throughout the fluid. The results presented here are identical with those of Kaplan.

G. N. Lance (Southampton).

Multhopp, H. Methods for calculating the lift distribution of wings (subsonic lifting-surface theory). *Aero. Res. Council, Rep. and Memo. no. 2884* (1950), 96 pp. (1955).

The integral equation for the load distribution on a lifting surface of given camber distribution in incompressible inviscid small-disturbance flow is written down. Its approximate solution is undertaken by combining a number of standard distributions and satisfying the equation at certain "pivotal points." The standard distributions selected are carried over from the "Glauert distributions" of two-dimensional airfoil theory. The spanwise integration of these distributions is carried out by the familiar methods of lifting-line theory; i.e., the distributions are approximated by trigonometric series. In principle, at least, the problem is then reduced to that

of satisfying simultaneously a number of algebraic equations.

The formulas worked out in detail here are for one and two standard distributions with one and two pivotal points at each spanwise station, respectively. The first of these gives equations of exactly the same form as in the author's well-known method for the lifting line [Luftfahrtforschung 15 (1938), 153-169], which are solved in the same manner. The more complicated case involves two equations, and again an iteration procedure is suggested. Two examples are worked out. In appendices are given detailed instructions for use of the method, special treatments of singular regions, and alternative derivations of some of the results.

W. R. Sears (Ithaca, N.Y.).

Hawthorne, W. R. The growth of secondary circulation in frictionless flow. *Proc. Cambridge Philos. Soc.* 51 (1955), 737-743.

From the Euler equations of steady, frictionless, adiabatic flow the author derives an expression for the derivative in the flow direction of the vorticity component parallel to the flow velocity. The presence of conservative body forces is included. This derivative denotes the rate of growth of secondary flow; i.e., the flow consisting of components normal to the general stream. Deductions from this equation are made for liquids, both uniform and stratified, perfect-gas flows with uniform stagnation pressure, perfect-gas flows with uniform stagnation temperature (as occur behind curved shock waves), and a perfect-gas atmosphere. The results are generalizations of results obtained earlier by Squire and Winter [J. Aero. Sci. 18 (1951), 271-277] and Hawthorne [Proc. Roy. Soc. London. Ser. A. 206 (1951), 374-387; MR 13, 177], which have been applied extensively to engineering problems.

W. R. Sears (Ithaca, N.Y.).

Hida, Kinzo. On some singular solutions of the Tricomi equation relating to transonic flow. *J. Phys. Soc. Japan* 10 (1955), 869-881.

Two types of singular solutions of the Tricomi equation for transonic flow are presented which have singularities corresponding to a uniform subsonic or supersonic flow at infinity upstream. The solutions tend to Guderley's solution [Hdqtrs. Air Materiel Command, Dayton, Ohio, Tech. Rep. no. F-TR-1171-ND (1948)] when the Mach number at infinity approaches unity. The solutions are given in the hodograph plane by an integral representation and their behavior in various parts of the hodograph plane, in particular near the singular point, is studied in terms of hypergeometric functions.

C. S. Morawetz.

Hida, Kinzo. Asymptotic behaviour of the location of a detached shock wave in a nearly sonic flow. *J. Phys. Soc. Japan* 10 (1955), 882-889.

The author shows that one of the solutions given in the preceding paper represents the flow behind a symmetrical shock with the proper behaviour at infinity. He deduces the relation, $b^{-1}R(1-M_{\infty}^2) = \text{constant}$, between the distance b of the shock from a body if there is one, its curvature R and the Mach number at infinity M_{∞} . The relations $b^{-1}\alpha(1-M_{\infty}^2)^2$ and $R^{-1}\alpha(1-M_{\infty}^2)^3$ are also determined. It is not clear to the reviewer how the position of the body is determined. The mapping into the physical plane is not discussed and the possibility of limiting lines is therefore not excluded. Some remarks are made about the axially symmetric case.

C. S. Morawetz (New York, N.Y.).

Manwell, A. R. Correction to my paper: "A new singularity of transonic plane flows". *Quart. Appl. Math.* 13 (1955), 337.

See *Quart. Appl. Math.* 12 (1955), 343-349 [MR 16, 535].

Martin, M. H. An existence and uniqueness theorem for unsteady one-dimensional anisotropic flow. *Comm. Pure Appl. Math.* 8 (1955), 367-370.

The flow behind a shock travelling down a shock tube after decay sets in is anisotropic, that is, the specific entropy varies from one fluid particle to another. In general the specific entropy will be known beforehand for each particle. This paper asks the questions, i) can an arbitrary curve (in the (x, t) -plane) with prescribed pressure variation along it be taken as the trajectory of a fluid particle in such a flow, and ii) if such a flow exists, is it unique? In other words, can the flow in the pipe be uniquely determined by singling out a gas particle and prescribing its motion and pressure? For a large class of curves it is shown that the answer is affirmative.

J. B. Serrin (Minneapolis, Minn.).

Roumieu, Charles. Sur la structure du choc oblique raccordant deux écoulements uniformes. *C. R. Acad. Sci. Paris* 241 (1955), 356-357.

It is shown that a uniform flow in the y -direction may be superposed upon a one-dimensional shock profile [cf. Gilbarg and Paolucci, *J. Rational Mech. Anal.* 2 (1953), 617-642; MR 15, 576], the result being a shock profile oblique to the flow directions at $\pm\infty$. *J. B. Serrin.*

Goldstine, Herman H., and von Neumann, John. Blast wave calculation. *Comm. Pure Appl. Math.* 8 (1955), 327-353.

This paper describes the method adopted for the computation, by means of an electronic computer, of the flow in a spherically symmetrical blast wave. The results are displayed in both tabular and graphical form.

D. C. Pack (Glasgow).

Mahony, J. J. A critique of shock-expansion theory. *J. Aero. Sci.* 22 (1955), 673-680, 720.

The assumption underlying the application of shock-expansion theory to the calculation of the pressure distribution on a thin sharp-nosed two-dimensional airfoil in a supersonic stream are examined, and it is suggested that they might be expected to lead to appreciable errors when the Mach Number is large. This case is then examined by the use of the "hypersonic analogy", and it is shown that for circular-arc airfoils the pressure distribution is given to good accuracy by shock-expansion theory even when the quantities neglected are no longer small. An asymptotic form for the decay of the leading-edge shock is developed in the case of shocks that are too strong initially for the Friedrichs theory to apply. (Author's summary.)

H. Cabannes (Marseille).

Fenain, Maurice, et Germain, Paul. Sur la résolution de l'équation régissant, en seconde approximation, les écoulements d'un fluide autour d'obstacles tridimensionnels. *C. R. Acad. Sci. Paris* 241 (1955), 276-278.

This paper is concerned with supersonic flow round a symmetrical delta wing with subsonic leading edges at zero incidence. As is well-known, the expansion of the velocity vector in terms of a (thickness) parameter leads, in the first approximation, to Laplace's equation (of two

variables) in a suitable system of cone field coordinates. It is shown here by an ingenious method that, similarly, the second-order terms can be found as solutions of Poisson's equation. The authors state that they have computed a number of numerical examples but that their results do not agree with the work of F. Moore [*J. Aero. Sci.* 17 (1950), 328-334, 383] and Tan. *A. Robinson.*

Fell, J., and Leslie, D. C. M. Second-order methods in inviscid supersonic theory. *Quart. J. Mech. Appl. Math.* 8 (1955), 257-265.

Let $\phi + \varphi$ be the perturbation velocity potential due to steady, inviscid, supersonic flow past a disturbing body, where ϕ is the first-order potential and satisfies $L(\phi) = \phi_{xx} + \phi_{yy} - (M^2 - 1)\phi_{zz} = 0$ obtained by neglecting all products of the potential, and φ is the second-order potential, neglecting all products involving φ , satisfies $L(\varphi) = \chi(\phi)$, where χ consists of double and triple products of ϕ and its derivatives. The authors obtain a particular integral for φ by neglecting the triple products in ϕ and superimposing fundamental solutions; they state that their technique also may be used to handle the triple products (which may be of the same order of magnitude as the double products in certain problems) but that the functions involved probably are very complex. The results are illustrated by rederiving Van Dyke's particular integral for axially symmetric flow [*J. Aero. Sci.* 18 (1951), 161-178, 216; MR 14, 331]. Future publication of results for yawed bodies of revolution and finite-aspect-ratio wings is promised. *J. W. Miles (Los Angeles, Calif.).*

Hunn, B. A. On the determination of the flutter forces on wings with supersonic leading edges. *Quart. J. Mech. Appl. Math.* 8 (1955), 293-310.

The velocity potential on the surface of a wing having straight, supersonic leading and trailing edges and a straight tip parallel to the line of flight, due to distortions that are harmonic in time and polynomials in the space coordinates, is expressed in terms of integrals of terms like $r^{2\theta} e^{-in\theta} (\cos 2\theta)^p (\sin 2\theta)^q$ over appropriate areas of an (r, θ) -plane. Evvard's method is used to handle the wing-tip effects; although the author recognizes that this is not valid unless terms of second order in the frequency are neglected, he implies that it often will prove sufficiently accurate for practical flutter problems not involving control surfaces. A numerical example is presented and the results compared with the first-order (in frequency) approximation obtained by neglecting terms of second order in frequency from the outset, in which case the problem transforms to one in steady flow. The reviewer notes that, since the error committed in applying Evvard's method is of the same order as that in the first-order approximation, the numerical comparison offered is somewhat misleading, especially in view of the author's implication that his results are of greatest value for low-aspect-ratio wings (two-dimensional theory being adequate for wings of high aspect ratio). *J. W. Miles.*

Fraenkel, L. E. Supersonic flow past slender bodies of elliptic cross-section. *Aero. Res. Council, Rep. and Memo. no. 2954* (1952), 27 pp. (1955).

This is a detailed application of slender-body theory [see Ward, *Quart. J. Mech. Appl. Math.* 2 (1949), 75-79; MR 10, 644] to slender bodies with elliptic cross-section. As far as the general method is concerned, the main innovation appears to lie in the application of the Joukowski transformation to convert the elliptic cross-

sections into circles in the transverse planes. The effect of discontinuities in the surface slope is considered [see Lighthill, *ibid.* 1 (1948), 90-102; MR 10, 76] for bodies of revolution. Formulae are obtained for drag, lift and pitching moment. The case of an elliptic cone is worked out in detail. In the limit this overlaps with a delta-wing considered by Squire [Aero. Res. Council, Rep. and Memo. no. 2549 (1951); MR 14, 109]. Good agreement between the two theories is obtained for this case, except in the neighbourhood of the leading edge. *A. Robinson.*

Kinber, B. E. Solution of the inverse problem of geometrical acoustics. *Akust. Z.* 1 (1955), 221-225. (Russian)

The author describes general principles for determining the reflecting surfaces or refracting lens necessary to convert one given acoustic field into another given acoustic field. He discusses specifically plane-wave and axisymmetric problems in which the rays are straight lines. The two given fields are related to each other by their geometries and by the law of continuity of energy in incident and reflected or refracted ray bundles. He shows that two reflections or two refractions are necessary for the conversion. *W. W. Soroka (Berkeley, Calif.).*

See also: Protter, p. 270; Teodorescu, p. 270; Ludford, p. 271; Hantush and Jacob, p. 271; Coburn, p. 297; Bodaszewski, p. 316.

Elasticity, Plasticity

Adkins, J. E. Some general results in the theory of large elastic deformations. *Proc. Roy. Soc. London. Ser. A.* 231 (1955), 75-90.

For finite elastic strain, this paper obtains a class of exact solutions which includes and generalizes a good many of those previously known. The technical nature of the work makes it difficult to describe in a review, but enough must be said to make it clear that this paper because of its inclusiveness is an important one for those attempting to solve particular problems of large strain. The analysis is not limited to isotropic bodies but concerns bodies whose aeolotropy, if any, is compatible with a particular class of deformations. In the author's words, "The analysis for each problem is performed initially for bodies possessing a suitable type of curvilinear aeolotropy, and results are derived which are independent of symmetries in the elastic material. These results are therefore valid, not only for the general type of material initially considered, but also for isotropic bodies and for materials which are orthotropic or transversely isotropic with respect to the curvilinear co-ordinate system which defines the aeolotropy. Both compressible and incompressible bodies are considered." The problems concern cylindrically symmetrical deformation of materials possessing a certain kind of cylindrical symmetry; extension, torsion, shear, and bending of annular segments or blocks and a generalized shear are included. The analysis is closely connected with a number of other recent studies of finite elasticity, cited by the author and too numerous to list here. That problems of this degree of generality are now being attacked successfully and compactly is evidence of the enormous progress made since 1948 in general elasticity, which is now an independent and developed branch of rational mechanics, worthy of a definitive mathematical treatise. *C. A. Truesdell.*

Mazzarella, Franco. Determinazione delle componenti di secondo ordine della deformazione riferite ad un generico sistema di coordinate curvilinee. *Rend. Accad. Sci. Fis. Mat. Napoli* (4) 21 (1954), 107-114.

The author gives the explicit forms of the components of Green's tensor of finite strain in orthogonal curvilinear coordinates, specializing them to cylindrical co-ordinates and to revolution co-ordinates. *C. A. Truesdell.*

Jindra, F. Reine ebene Biegung bei einem nichtlinearen Elastizitätsgesetz. *Ing.-Arch.* 23 (1955), 373-378.

Assuming a special form of relations giving infinitesimal strain in terms of stress, the author solves two problems in nonlinear elasticity. These are for bending of rectangular and circular blocks by terminal couples. *J. L. Ericksen.*

Jindra, F. Der ebene Verzerrungszustand des dickwandigen Rohres bei einem nichtlinearen Elastizitätsgesetz. *Ing.-Arch.* 23 (1955), 122-129.

The author continues his apparent program of obtaining complicated series solutions according to a special non-linear theory for problems for which a simple solution according to the general theory of finite elastic strain is known. [For his prior work and criticism of it, see *Ing. Arch.* 22 (1954), 121-144; 411-418; MR 16, 88, 765.] The problem treated here is the hollow cylinder subject to uniform inside and outside pressure; the general solution, as for the problem treated in the author's paper reviewed above, was obtained by A. E. Green [*Proc. Roy. Soc. London. Ser. A.* 227 (1955), 271-278; MR 16, 764] in a paper not available when the author wrote; the line of attack, however, follows closely the prior solution by Rivlin for incompressible bodies [*Philos. Trans. Roy. Soc. London. Ser. A.* 242 (1949), 173-195; MR 11, 627].

C. A. Truesdell (Bloomington, Ind.).

Bordoni, Piero Giorgio. Limitazioni per gli invarianti di deformazione. *Rend. Mat. e Appl.* (5) 14 (1955), 269-279.

Assume $c_1 \geq 0$ and put $I = c_1 + c_2 + c_3$, $II = c_1c_2 + c_1c_3 + c_2c_3$, $III = c_1c_2c_3$. Then

- $27III \leq I^3$, equality if and only if $c_1 = c_2 = c_3$;
- $12I \leq (3+I)^2$, equality if and only if $I = 3$.

Both (a) and (b) were observed by Signorini [*Ann. Mat. Pura Appl.* (4) 30 (1949), 1-72; see Ch. II, § 5; MR 11, 756], who concluded

- $18^2III \leq I^3(3+I)^2$, equality if and only if $c_1 = c_2 = c_3 = 1$.

It is obvious that (c) is inferior to (a), which has evident interest for finite strains because it affords both upper and lower bounds for the change of volume in terms of the traces of the deformation tensors C and c . The author, although he cites Signorini's paper, appears to have noticed only (c), since he gives a complicated derivation resting on continuity considerations and occupying several pages. He also describes the surface of values of I and II which can correspond to a fixed value of III . Both Signorini and the author succeed in making the results appear complicated by expressing it in the notations favored by the Italian school.

The author applies (a) to show that a certain restriction he had previously imposed on a certain particular strain energy is automatically satisfied. *C. A. Truesdell.*

Kroupa, František. Plane deformation in the non-linear theory of elasticity. Czechoslovak J. Phys. 5 (1955), 18-29. (Russian summary)

The author adopts the Mooney-Rivlin theory of large elastic deformations of an incompressible material. He sets up and solves the equations for torsional shearing of a circular cylinder. Apparently he is not aware that Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. 242 (1949), 173-195; MR 11, 627] has obtained the solution to a more general problem for general strain energy and has discussed the specialization of that case to the one treated by the author [§ 13 of Rivlin's paper]. The author works out numerical results corresponding to parameter values he considers practically appropriate and concludes that the maximum tension and shear stress exceed the predictions of the classical linear theory by 300% and 100% respectively. C. A. Truesdell (Bloomington, Ind.).

Nowiński, Jerzy. Some selected problems of the theory of heterogeneous elastic bodies. Arch. Mech. Stos. 6 (1954), 665-692 (1955). (Polish. Russian and English summaries)

The effect is investigated of the non-homogeneity of the elastic medium as specified by certain linear and non-linear laws of variation with the coordinates of the elastic modulus, on the stress- and displacement-field in a number of particular problems, such as the steady state vibration of a one-dimensional elastic beam, the buckling of a homogeneous elastic strut within a non-homogeneous elastic medium, the thick-walled cylinder with external pressure and the concentrated force acting on the inhomogeneously elastic half-space. In all but the first case the resulting differential equations with variable coefficients can not be rigorously solved, and variational methods have been used. A. M. Freudenthal.

Cristea, M. La loi de Hooke plane non isotrope. Com. Acad. R. P. Române 1 (1951), 1007-1012. (Romanian. Russian and French summaries)

The author notes the specialization of the strain energy in linear elasticity corresponding to the different crystal types in a strictly plane theory. C. A. Truesdell.

Bodaszeński, Stanisław. On the asymmetric state of stress and its applications to the mechanics of continuous mediums. Arch. Mech. Stos. 5 (1953), 351-396. (Polish. Russian and English summaries)

The paper is an analysis of the elastic state of stress and strain based on the assumption that the stress tensor under conditions of equilibrium is non-symmetrical, and that its asymmetric part, the components of which are $\frac{1}{2}(\tau_{jk} - \tau_{kj}) = \theta_i \neq 0$, is in equilibrium with a postulated "dipolar moment of internal surface forces" D_i , presumably caused by internal couples, and is related to the asymmetric tensor ω_i of rigid-body rotation; this relation is arbitrarily assumed to be linear. The so-called non-symmetrical state of stress and strain is therefore governed by two additional equations, an equilibrium condition $\theta_i + D_i = 0$ and a constitutive equation $\theta_i = H\omega_i$, where H is a new physical constant. The potential energy of the system has an additional term $\frac{1}{2}\theta_i\omega_i$.

The application of the concept of a non-zero θ -stress is extended to the hydrodynamics of the so-called "polar" liquid and an attempt is made to prove that a critical value of the θ -stress delimits conditions of laminar and turbulent flow.

Although the underlying assumption of the theory are

purely speculative, to say the least, the (academic) value of the paper is in its discussion of the results arising from such an assumption. A. M. Freudenthal.

Stahl, K. Über die Lösung ebener Elastizitätsaufgaben in komplexer und hyperkomplexer Darstellung. Ing.-Arch. 22 (1954), 1-20.

The methods for the solution of plane elasticity problems referred to in the title are: (1) the method based on the representation of the stress components and the displacements in terms of analytic functions $f(x+iy)$ of a single complex variable $x+iy$, where $1+i^2=0$, which goes back to G. Kolosoff [Z. Math. Phys. 62 (1914), 384-409] and N. Muschelišvili [Math. Ann. 107 (1933), 282-312; see also Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; MR 11, 626; 15, 370; and A. E. Green and W. Zerna, Theoretical elasticity, Oxford, 1954; MR 16, 306]; (2) the method based on the representation of the stress components and the displacements in terms of analytic functions $f(x+jy)$ of a hypercomplex variable $x+jy$, where $(1+j^2)=0$, which goes back to L. Sobrero [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 6 (1934), 1-64; Theorie der ebenen Elastizität, Teubner, Leipzig, 1934] and has been recently studied by K. Schmidt [Ing.-Arch. 19 (1951), 324-341; MR 14, 428]. In the present paper the author studies the relationships between these two methods, giving transformation formulas which enable one to pass readily from one representation to the other, and applies both procedures to many particular problems, determining the suitability of each method to the individual cases considered.

J. B. Diaz (College Park, Md.).

Grioli, Giuseppe. Sull'equilibrio dei corpi elastici. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 371 (1953), 5 pp.

This short note contains some remarks concerning application to linear elasticity of the author's theorems on mean values of the products of stresses by orthogonal polynomials [Ann. Mat. Pura Appl. (4) 33 (1952), 239-246; MR 14, 514]. He states without proof a lower bound for the variation of temperature in an adiabatic thermo-elastic deformation. Extending his earlier work on St. Venant's problem for crystalline bodies [Rend. Sem. Mat. Univ. Padova 21 (1952), 228-242; MR 14, 701], he obtains a formula for the stresses in terms of certain means and of the limits of the solutions of certain minima of means. C. A. Truesdell (Bloomington, Ind.).

Hill, R. On related pairs of plane elastic states. J. Mech. Phys. Solids 4 (1955), 1-9.

The author studies problems of plane strain in an incompressible elastic body and shows that for the known solution of a boundary-value problem in which stresses are prescribed the solution of a different problem may be obtained. Also, a boundary-value problem defined by surface displacements can be converted into one defined by surface tractions, or vice-versa. Analogous results are stated for steady quasi-static viscous flow.

[By considering known expression for displacements, stresses and resultant force for plane strains or generalized plane stress in terms of complex potentials $f(z)$, $g(z)$, more general results may be obtained which are valid for compressible or incompressible materials. Suppose $h(z) + z f'(\bar{z}) + \bar{z} g'(\bar{z})$ takes prescribed values on the boundaries where k is a constant. Also, if stresses and displacements

are single-valued,

$$\kappa[f(z)] = [\bar{g}'(\bar{z})], [f'(z)] = [g''(z)] = 0,$$

where $[\]$ denotes the change in value of the function in traversing a complete contour inside the body, $\kappa = 3 - 4\sigma$ for plane strain and $\kappa = (3 - \sigma)/(1 + \sigma)$ for generalized plane stress, σ being the Poisson ratio. Solutions to problems for either prescribed stresses or prescribed displacements at the boundaries may then be found by giving h the values 1 and $-\kappa$ respectively.]

A. E. Green.

★Schumann, Walter. *Theoretische und experimentelle Untersuchungen über das Saint-Venantsche Prinzip, speziell mit Anwendung auf die Plattentheorie*. Dissertation, Eidgenössische Technische Hochschule in Zürich, 1955. 85 pp.

This dissertation consists of a theoretical and experimental part, both concerned with Saint-Venant's principle in the theory of elasticity. The present review is confined to the theoretical portion, which aims at proofs of several theorems announced by the author in a previous note [C.R. Acad. Sci. Paris **238** (1954), 988-990; MR **15**, 840]. Some of the historical inaccuracies pointed out in the review of the earlier note, have been corrected. Nevertheless, the author continues to ignore von Mises' observation [Bull. Amer. Math. Soc. **51** (1945), 555-562; MR **7**, 40] that the entire Boussinesq formulation of Saint-Venant's principle fails for bounded domains. The author's chief objective is to strengthen the von Mises formulation of the principle for which a general proof was given by the reviewer [Quart. Appl. Math. **11** (1954), 393-402; MR **15**, 370]. Unfortunately, the author succeeds primarily in spreading further confusion on a subject which has already displayed an unusual degree of tolerance toward conceptual vagueness and careless reasoning.

A discussion of one of the author's new theorems ought to substantiate the foregoing claim. Consider an elastic half-space $z \geq 0$ and let $\xi_i(\epsilon)$, $\eta_i(\epsilon)$ ($i = 1, 2, \dots, n$) be cartesian coordinates of a one-parameter family of points $Q_i(\epsilon)$ in the plane $z = 0$, such that $\xi_i, \eta_i \rightarrow 0$ as $\epsilon \rightarrow 0$. Let \mathbf{F}_i (independent of ϵ) be a set of concentrated loads applied at the points $Q_i(\epsilon)$. Let $\sigma(\epsilon, P)$ be a component of stress, induced by the given loading, at an interior point P which has a distance r from the origin. The author now asserts: if $\sum_{i=1}^n \mathbf{F}_i = 0$, then $\sigma = O(\epsilon r^{-3})$. This statement is, to begin with, obscure. If what is intended is $\sigma = O(\epsilon r^{-3})$ as $\epsilon \rightarrow 0$ for fixed r , then the statement is a trivial consequence of von Mises' formulation of the principle, according to which here $\sigma = O(\epsilon)$ (or smaller) as $\epsilon \rightarrow 0$ for fixed r ; indeed in this case $\sigma = O(\epsilon r^\alpha)$ follows, where α is arbitrary. What the author evidently means is: $\sigma = O(\epsilon)$ as $\epsilon \rightarrow 0$ for fixed r and $\sigma = O(r^{-3})$ as $r \rightarrow \infty$ for fixed ϵ . The second part of this statement may be inferred from an elementary computation based on the known solution for the half-space under concentrated loads. It is deduced by the author with the aid of an unnecessary restriction on the ϵ -family of loadings and by means of a fallacious equilibrium argument.

This and similar propositions (similarly stated and proved) the author seeks to extend to bounded domains in a second group of theorems. The extension is to be accomplished through the introduction of "correction terms" for the corresponding order-of-magnitude estimates appropriate to unbounded regions. Since $r \rightarrow \infty$ has no meaning in connection with a finite body, the reviewer is unable to follow the remaining developments.

On p. 29, the author claims to show that von Mises'

version of Saint-Venant's principle is not applicable to Saint-Venant's problem; his proof is based on a misinterpretation of the modified principle. E. Sternberg.

Mossakovs'kiĭ, V. I., and Zagubiženko, P. A. *On compression of an elastic isotropic plane weakened by a rectilinear gap*. *Dopovidi Akad. Nauk Ukrain. RSR* **1954**, 385-390. (Ukrainian. Russian summary)

The problem stated in the title is solved by the Muskhelishvili method for the case when the width of the gap is negligible and for the case when the width is of the order of magnitude of elastic deformations.

I. S. Sokolnikoff (Los Angeles, Calif.).

Goodier, J. N., and Wilhoit, J. C., Jr. *Axial displacement dislocations for the hollow cone and the hollow sphere*. *Quart. Appl. Math.* **13** (1955), 263-269.

As indicated in the title of the paper simple solutions in closed form are given for the dislocational states of stress in a hollow cone or a hollow sphere induced by making an axial cut, and imposing a rigid-body displacement of one force of the cut relative to the other in the axial direction. The problems are solved by adaptations of the torsion theory of B. de Saint-Venant and of J. H. Michell's theory of torsion of non-uniform shafts.

R. Gran Olsson (Trondheim).

★Dinnik, A. N. *Prodol'nyi izgib. Kručenje. [Longitudinal bending. Torsion.]* Izdat. Akad. Nauk SSSR, Moscow, 1955. 392 pp. 18.80 rubles.

This volume consists of a reproduction of previously published work by A. N. Dinnik on longitudinal bending of rods and on torsion of beams. It is divided into three parts. The first of these, consisting of 209 pages, is a reproduction of the author's monograph "Longitudinal bending (theory and applications)" [GONTI, Moscow, 1939]. It is intended primarily for those concerned with engineering design. Most results are presented either in tabular form or as formulas suitable for practical use. The exposition of the theory is illustrated by numerous examples. This part is divided into seven chapters, each followed by a list of sources. The chapter headings are: (1) Critical load. (2) Longitudinal bending of straight rods of uniform cross-section. (3) Rods of variable cross-section. (4) Longitudinal bending under distributed load. (5) Longitudinal bending of curvilinear rods. Circular arc. Ring. (6) Longitudinal bending of curvilinear rods. Parabolic and chain arcs. Helical spring. (7) Longitudinal bending of systems of rods.

The second part (pp. 213-255) consists of three supplementary papers on stability of rods, written by Dinnik after 1939. They are: (1) Stability of quite shallow arcs. (2) The effect of elastic fixing of ends on the stability of compressed rods. (3) Stability of rods of variable cross-section under stresses beyond the proportional limit.

The third part (pp. 259-387) is a reproduction of the author's small book "Torsion (theory and applications)" [GONTI, Moscow, 1938], likewise intended primarily for engineers. It consists of five chapters: (1) Introduction (consisting of an outline of the basic theory). (2) Methods (concerned with different techniques of solving Saint-Venant's torsion problem). (3) Principal results (including a brief discussion of the torsion of beams of several particular cross-sections). (4) Shaft of variable cross-section. Effect of fixing cross-sections. Curvilinear rods. (5) Vibrations.

This volume does not impose severe demands on the

reader's mathematical equipment and will be welcomed by engineers. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Barta, J. Sur l'estimation de la rigidité de torsion des prismes multicellulaires à parois minces. *Acta Tech. Acad. Sci. Hungar.* 12 (1955), 333-338. (Russian, English and German summaries)

For a beam whose cross-section consists of thin walls enclosing n empty cells the torsion problem reduces approximately to the simultaneous linear equations

$$TI_i + \sum_{j=1}^n (T_i - T_j)I_{ij} = 2\Omega_i \quad (i=1, \dots, n).$$

Here I_i , I_{ij} , and Ω_i are positive geometric constants of the beam and $I_{ii}=I_i$. The unknowns T_i are easily seen to be all positive. The torsional rigidity is given by

$$R = \sum_{i=1}^n 2\Omega_i T_i.$$

The author proves the following theorem on linear equations. Let T_i^* be any set of positive constants and let k and K be the minimum and maximum, respectively, of the ratios

$$(T_i^* I_i + \sum_{j=1}^n (T_i^* - T_j^*) I_{ij}) / (2\Omega_i) \quad (i=1, \dots, n).$$

Then

$$K^{-1} \sum 2\Omega_i T_i^* \leq R \leq k^{-1} \sum 2\Omega_i T_i^*.$$

(Although it is not stated, K and k must, of course, be assumed positive.) If T_i^* is a good approximation to T_i , the numbers K and k are both near one, and close upper and lower bounds for R are obtained.

H. F. Weinberger (College Park, Md.).

Lin, Hung-Sun. On variational methods in the problem of torsion for multiply-connected cross sections. *Acta Sci. Sinica* 3 (1954), 171-186.

The Saint-Venant torsion problem for a prismatic body with simply or multiply connected cross-section is discussed by starting from the basic variational principles of minimum potential energy and minimum complementary energy of elasticity (in this connection, cf. a brief note by C. Weber [*Z. Angew. Math. Mech.* 11 (1931), 244-245] which is not mentioned by the author). A few numerical examples are given for illustration, and the results are compared with those of S. P. Timoshenko [*Theory of elasticity*, McGraw-Hill, New York, 1934, p. 270] and J. B. Diaz and A. Weinstein [*Amer. J. Math.* 70 (1948), 107-116; MR 9, 480; cf. also the examples in the paper of Weber cited above]. *J. B. Diaz.*

Babakova, O. I. On torsion of bars with Z-shaped cross-section. *Dopovidi Akad. Nauk Ukrain. RSR* 1954, 319-323. (Ukrainian. Russian summary)

Saint-Venant's torsion problem for a bar with an infinite Z-shaped cross-section is solved by the method of E. Trefftz [*Math. Ann.* 82 (1920), 97-112].

I. S. Sokolnikoff (Los Angeles, Calif.).

Šerman, D. I. Torsion of a circular cylinder stiffened with an elliptic bar. *Inžen. Sb.* 21 (1955), 79-96. (Russian)

The state of stress in a twisted circular cylinder reinforced by an elliptic rod with coincident axis is determined by reducing the solution of the Saint-Venant torsion problem to an infinite system of linear algebraic equations. This system is shown to be completely regular, and its

solution is illustrated by a numerical example. The corresponding problem for an elliptical rod reinforced by a circular cylinder has been solved by the author [*Inžen. Sb.* 10 (1951), 81-108; MR 15, 372]. *I. S. Sokolnikoff.*

Amen-zade, Yu. A. Bending of a circular prismatic beam with an elliptic cavity. *Inžen. Sb.* 21 (1955), 97-112. (Russian)

A device suggested by Šerman in the paper reviewed above is used to solve the problem of bending a cantilever beam, described in the title, by a load parallel to the minor axis of the ellipse. The paper is illustrated by detailed calculations. *I. S. Sokolnikoff.*

Čobanyan, K. S. Application of the stress function in the problem of torsion of prismatic bars composed of several materials. *Akad. Nauk Armyan. SSR Izv. Fiz.-Mat. Estest. Tehn. Nauki* 8 (1955), no. 2, 17-30. (Russian. Armenian summary)

The Saint-Venant torsion problem for compound beams is phrased in terms of Prandtl's stress function and illustrated by a solution of the problem for a beam composed of two rectangular beams so welded as to form a T-section. *I. S. Sokolnikoff* (Los Angeles, Calif.).

de Schwarz, Maria Josepha. Sulla torsione dei prismi cavi regolari. *Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo* no. 371 (1953), 4 pp.

The author applies the method of Fichera [*Rend. Mat. e Appl.* (5) 12 (1953), 163-176; MR 15, 758] for solving the torsion problem for a cylinder whose cross-section is a polygonal annulus. The author considers the case when the two polygons are regular, similar, concentric, with parallel sides, and with radii 1 and R . She proves that given any R_1 and R_2 such that $1 < R_1 < R_2$, there exists a number $\mu(R_1, R_2)$ such that for any R between R_1 and R_2 and for polygons whose number of sides exceeds μ the first derivatives of the torsion function are infinite at the internal corners. *C. A. Truesdell* (Bloomington, Ind.).

Reissner, Eric. On some aspects of the theory of thin elastic shells. *J. Boston Soc. Civil Engrs.* 42 (1955), no. 2, 100-133.

Following a brief review of the statical equations of the linear membrane theory of shells in the form given by Pucher, and their reduction (in terms of the Airy stress function) to a single differential equation, the author illustrates a method for obtaining a solution of this differential equation by expansion in terms of a non-dimensional rise of the shell; this leads to results suitable for shallow membranes. Next, by application of the principle of minimum complementary energy, a system of suitable stress-strain relations (involving pseudo-stress resultants) are deduced for the linear membrane theory and reduced systematically to those appropriate for shallow membranes. In the second part of the paper Marguerre's equations for shallow shells (which include bending and non-linear effects) are reduced to two simultaneous differential equations for an Airy stress function and axial displacement. These equations are applied to an example of an hyperbolic paraboloid shell under its own weight to study the possible buckling of such a shell and the effect of support conditions on the state of stress.

The paper, written in a lucid fashion, is expository in character and contains a number of additions to the existing literature on the subject. *P. M. Naghdi.*

Sokolov, A. M. On the region of applicability of the momentless theory to the computation of shells of negative curvature. *Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk* 1955, no. 5, 85-101. (Russian)

This paper treats the theory of elastic deformations of membrane shells in equilibrium. It is supposed that the middle surface is part of a hyperboloid of one sheet. Attention is given primarily to a general problem involving such a shell with two plane ends taken normal to the principal axis lying entirely within the hyperboloid. This shell is allowed to undergo certain deformations subject to end constraints. Certain restrictions on the solution are discussed in detail. *H. G. Hopkins.*

Vlasov, V. Z. On the theory of momentless shells of revolution. *Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk* 1955, no. 5, 55-84. (Russian)

This paper treats the general theory of elastic membrane shells of revolution in equilibrium. Specific problems are not considered. Attention is directed principally towards the case when there is no surface loading. The homogeneous system of equations for the problem is solved identically through the introduction of stress and displacement functions. A detailed study is made of these functions for the cases of elliptic, hyperbolic and some other types of shell. The paper concludes with a study of a shell formed from one or more conical frustums and with or without reinforcing rings. *H. G. Hopkins.*

Berger, E. R. Ein Minimalprinzip zur Auflösung der Plattengleichung. *Österreich. Ing.-Arch.* 7 (1953), 39-49.

The minimum-potential-energy principle singles out, from among all functions satisfying the boundary conditions (save the free-boundary conditions) of a plate problem, that function which represents the actual displacement. Dirichlet's principle plays a similar rôle for the Dirichlet problem for the Laplace differential equation, and E. Trefftz [Verh. 2. Internat. Kongresses Tech. Mech., Zürich, 1926, Füssli, Zürich, 1927, pp. 131-137] has derived a corresponding maximum principle for the Dirichlet problem which isolates the solution of a Dirichlet problem among the class of functions satisfying the Laplace differential equation [cf. K. O. Friedrichs, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1929, 13-20; Courant and Hilbert, Methods of mathematical physics, vol. I, Interscience, New York, 1953, pp. 231-257; MR 16, 426]. The author follows this mentioned procedure of Trefftz to derive a maximum principle which picks out, from among all functions satisfying the differential equations and the free-boundary conditions of a plate problem, that function which represents the actual displacement. *J. B. Diaz* (College Park, Md.).

Vorob'ov, L. N. On a polynomial solution of the plane problem for a rectangular orthotropic plate. *Dopovidi Akad. Nauk Ukrain. RSR* 1954, 391-394. (Ukrainian. Russian summary)

A polynomial solution of the plane elastostatic problem for an orthotropic rectangular plate, two parallel edges of which are subjected to the action of prescribed forces, is obtained by a method of successive approximations.

I. S. Sokolnikoff (Los Angeles, Calif.).

Huth, J. H. Thermal stresses in a partially clamped elastic half-plane. *J. Appl. Phys.* 23 (1952), 1234-1237.

With the half plane $y \leq 0$, and the clamping at $y=0$,

$-1 \leq x \leq 1$, the problem is solved by Muskhelishvili's method using complex functions. Closed form expressions are given for stress components and displacements which are also graphed.

E. H. Lee (Providence, R.I.).

Rüdiger, D. Die strenge Theorie anisotroper prismatischer Faltwerke. *Ing.-Arch.* 23 (1955), 133-150.

This paper treats in detail the satisfaction of appropriate compatibility conditions along the junctions of composite stiffened anisotropic plates in the analysis of small elastic deformations corresponding to impressed load distributions. *F. B. Hildebrand.*

Schnell, W. Berechnung der Stabilität mehrfeldriger Stäbe mit Hilfe von Matrizen. *Z. Angew. Math. Mech.* 35 (1955), 269-284. (English, French and Russian summaries)

The determination of the Euler load for beams with piecewise constant bending stiffness, subject to various conditions of support at the ends and at intermediate points, is systematized by matrix methods. A tabulation of relevant auxiliary functions is included.

F. B. Hildebrand (Cambridge, Mass.).

Robinson, J. R. The buckling and bending of orthotropic sandwich panels with all edges simply-supported. *Aero. Quart.* 6 (1955), 125-148.

The author deals with buckling in compression of an orthotropic rectangular sandwich pane l with simply-supported edges, subject also to a distribution of lateral loading, under the assumption that the panel can be simulated by a plate characterized by two flexural stiffnesses, a twisting stiffness, two transverse shear stiffnesses, and two Poisson moduli. Limiting cases are analyzed numerically. *F. B. Hildebrand.*

Skuridin, G. A. Approximate solution of the problem of diffraction of a plane elastic wave relative to a crack. *Izv. Akad. Nauk SSSR. Ser. Geofiz.* 1955, 3-16 (1955). (Russian)

The author gives an approximate solution to the problem of the diffraction of plane longitudinal elastic waves with respect to a crack by means of the generalized Huyghens-Kirchhoff principle (as applied to the dynamical equation of motion) [cf. V. D. Kupradze, Boundary problems of the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950; MR 15, 318; Love, A treatise on the mathematical theory of elasticity, 4th ed., Cambridge, 1927]. The details of the analysis are too complicated to be reproduced here. Additional information, closely related to the author's investigation, can be found in Scherman, C.R. (Dokl.) Acad. Sci. URSS (N.S.) 48 (1945), 626-629 [MR 8, 120] and Kupradze, Fundamental problems of the mathematical theory of diffraction, Moscow, 1935 [cf. MR 14, 877 for a translation].

The crack is situated symmetrically with respect to the origin on the y -axis ($+a, -a$). The components u and v of the displacement vector are given in closed integral (complicated) forms, for which approximate expressions are obtained. The final results are exemplified by numerous curves for different angles of the impinging wave; some of the curves show sharp resonances. The results seem to be reasonable. *K. Bhagwandin* (Oslo).

Zoller, K. Über die Koppelung der Dehnungs- und Torsionsschwingungen von umlaufenden Scheiben. *Ing.-Arch.* 23 (1955), 254-261.

The vibrations of a revolving elastic disk in the shape of an annulus are studied. The equations of motion are derived using Hamilton's principle. In the case of harmonic oscillations the author applies the Rayleigh-Ritz method, obtaining approximations to the vibration frequencies. A numerical example is carried out in detail.

E. Pinney (Berkeley, Calif.).

Saibel, Edward, and Lee, Winston F. Z. Vibrations of a continuous beam under a constant moving force. *J. Franklin Inst.* 254 (1952), 499-516.

Continuous beams with rigid or elastic internal supports are considered. The motion is expressed in terms of the known eigen-functions of the beam without internal supports. The corresponding generalized coordinates satisfy the Lagrange's equations for the system, the internal support conditions, and the applied loads. Explicit formulae are given for a single internal support and a sinusoidally varying point force. A moving force is treated by its Fourier series, each sinusoidally distributed component being treated by integration from the point force case. Examples are given which agree with earlier work using a less general approach. It is claimed that this method is much easier to apply than the standard treatments.

E. H. Lee (Providence, R.I.).

Aggarwal, Ram Ratan. Axially symmetric vibrations of a finite isotropic disk. I. *J. Acoust. Soc. Amer.* 24 (1952), 463-467.

Elastic vibration equations in cylindrical coordinates are used to provide special solutions by separation of variables. These are sufficiently general to satisfy zero traction boundary conditions on the plane surfaces normal to the axis, $z = \pm c$, but to satisfy zero traction on the cylindrical surface $r = a$ only approximately. Corresponding natural frequencies are determined and compared with compressional and shear thickness frequencies for a disc of infinite radius. The residual traction remaining at the "free" cylindrical surface is computed, and is compared with the axial stress at the center of the disc. In some cases it is small by comparison, in others not so. Thus while the method provides a good approximation to some modes, others are only poorly approximated.

E. H. Lee (Providence, R.I.).

Šatašvili, S. H. Reduction of a mixed problem of the theory of steady elastic vibrations to a Fredholm integral equation. *Soobšč. Akad. Nauk Gruzin. SSR* 14 (1953), 257-260. (Russian)

The boundary-value problem referred to consists in the determination of the longitudinal and transverse potential functions $\varphi(x, y)$ and $\psi(x, y)$ satisfying the following differential equations in a bounded, plane, simply connected, open set S :

$$\Delta\varphi + k_1^2\varphi = 0, \quad \Delta\psi + k_2^2\psi = 0,$$

(where Δ is the Laplacian, $k_1 = \omega/a_1$, $k_2 = \omega/a_2$, ω is the frequency of vibration, and a_1 and a_2 are respectively the speed of propagation of longitudinal and transverse waves) subject to certain mixed boundary conditions on the smooth boundary curve L . N. I. Mushelišvili [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad,

1949; MR 11, 626; 15, 370] first solved this problem when $\omega = 0$ and S can be conformally mapped, by a rational function, on the unit circle; the solution in the general case was given by D. I. Šerman [Prikl. Mat. Meh. 7 (1943), 413-420; MR 6, 195]. In the present paper, following an idea of Šerman's, the author constructs a system of "potentials" which enables him to reduce the solution of the boundary-value problem to that of Fredholm integral equations.

J. B. Diaz (College Park, Md.).

Pekeris, C. L. The seismic buried pulse. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 629-639.

In a previous paper [same Proc. 41 (1955), 469-480; MR 17, 213] the author considered a special type of point source pulse within a semi-infinite elastic medium. His results, while physically reasonable, were derived by a formal operational method whose validity was not apparent. In the present paper he seeks to transform these results to a more satisfactory form. This involves a transformation of the contour of integration of the integrals for the displacements and a formidable reduction of the new integrals, making use of the same operational methods.

E. Pinney (Berkeley, Calif.).

Das Gupta, Sushil Chandra. Propagation of Rayleigh waves in transversely isotropic medium in three dimensions. *Trans. Amer. Geophys. Union* 36 (1955), 675-678.

A crystalline medium with a vertical axis of symmetry and transverse isotropy is considered. The equations of elastic vibration are written down and separated in the harmonic, axially symmetric case. A wave that may be identified with Rayleigh's is found.

E. Pinney.

Bieniek, M. Principles of dynamics of non-elastic bodies. *Arch. Mech. Stos.* 4 (1952), 43-92 (1953). (Polish. English summary)

The paper establishes and analyzes with the help of Laplace transforms the equations of motion of compressible linear visco-elastic media the constitutive equations of which do not contain time derivatives of the stress- and strain-deviators of higher than second order. The resulting integro-differential equations for the displacement components are integrated for the problems of plane, cylindrical and spherical waves in the infinite medium, for the steady-state vibrations of systems on one degree of freedom, for the longitudinal and torsional vibrations of a one-dimensional strut as well as for longitudinal impact.

A. M. Freudenthal.

Moisi, G. C. Interprétation du corps des quotients d'opérateurs différentiels avec une application à la théorie des corps visco-élastiques. *Acad. Repub. Pop. Romine. Bul. Ști. Sec. Ști. Mat. Fiz.* 7 (1955), 127-138. (Romanian. Russian and French summaries)

The author applies theorems on fields of differential operators A, B to linear one-dimensional visco-elasticity. The results amount to simple remarks on linear differential equations with constant coefficients. From integrals of the system

$$A_1\psi = B_1\phi_1, \quad A_2\psi = B_2\phi_2,$$

integrals of

$$(A_1 + A_2)\psi = (B_1 + B_2)\phi$$

can be constructed.

C. A. Truesdell.

Mattice, H. C., and Lieber, Paul. On attenuation of waves produced in visco-elastic materials. *Trans. Amer. Geophys. Union* 35 (1954), 613-624.

The paper studies the wave motion produced when a pressure pulse is applied to the interior surface of a spherical cavity (radius δ) in an infinite, homogeneous, isotropic medium with viscoelastic properties (i.e. with the properties of a Kelvin solid). The displacement is directly proportional to pressure and δ , and inversely proportional to rigidity. At relatively large distances r from the origin only an oscillatory displacement is found, the amplitude of which decreases with $1/(r/\delta)$. Comparison with the purely elastic case shows that viscosity reduces the amplitude of the displacement in the vicinity of the wave front and the velocity of propagation of the wave. The problem applies to the propagation of waves in the Earth.

B. Gross (Rio de Janeiro).

Thomas, T. Y. On the structure of the stress-strain relations. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 716-720.

Truesdell [J. Rational Mech. Anal. 4 (1955), 83-133; MR 16, 880] has emphasized that in theories of the form (I) rate of stress = f (rate of strain, stress), the "rate of stress" must have the correct tensorial character. Truesdell devised a procedure which led to a definition of "rate of stress" in terms of the total time derivative of the components of the stress tensor, together with additional terms involving components of stress and the velocity gradients. The present author reconsiders the definition of "rate of stress" using rectangular coordinate systems and arrives at a result similar to that obtained by Noll [ibid. 4 (1955), 3-81; MR 16, 764]. The result differs from that given by Truesdell but only by additional terms which in a general theory of type (I) can be absorbed in the right-hand member of this relation. A. E. Green.

Thomas, T. Y. Combined elastic and Prandtl-Reuss stress-strain relations. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 720-726.

This paper is based on the assumption of a set of constitutive relations of the form

$$(I) \quad \text{rate of stress} = f(\text{rate of strain, stress})$$

in which the right-hand side are components of a tensor invariant of the stress tensor and rate-of-strain tensor, and the left-hand side is defined as in the paper reviewed above. Relations (I) are constructed so as to obtain the utmost simplicity consistent with assumptions of isotropy and homogeneity and they embrace both the elastic and plastic domain as commonly understood. A von Mises criterion of the form $\sigma_{\alpha\beta}^* \sigma_{\alpha\beta}^* \leq K$, where $\sigma_{\alpha\beta}^*$ is deviatoric stress, is assumed to hold throughout the motion, the extreme case of "yield" corresponding to equality. In this extreme case the constitutive relations become identical with the condition of incompressibility and the Prandtl-Reuss relations. [Although reference is made to loading and unloading in the introduction this question does not seem to be discussed later. Related work, which admittedly has deficiencies, is not mentioned. See, e.g., W. Prager, *Duke Math. J.* 9 (1942), 228-233; and G. H. Handelman, C. C. Lin, and W. Prager, *Quart. Appl. Math.* 9 (1947), 397-407; MR 4, 61; 9, 120]. A. E. Green.

Thomas, T. Y. Combined elastic and von Mises stress-strain relations. *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 908-910.

In the paper reviewed second above the author obtained

stress-strain relations

$$a \frac{D\sigma_{ii}}{Dt} = \epsilon_{ii} + b\sigma_{ii}, \quad A \frac{D\sigma_{\alpha\beta}^*}{Dt} = \epsilon_{\alpha\beta}^* + B\sigma_{\alpha\beta}^*,$$

where a , b , A , and B are scalar invariants of the stress tensor σ and the rate of strain tensor ϵ , and where D/Dt denotes the absolute time derivative (see the paper reviewed above). The "star" denotes deviatoric components of stress and rate of strain. Making clearly stated assumptions, the author shows how the invariants a , b , A , and B can be chosen so that the above equations become the stress-strain relations for incompressible flow of the von Mises type whenever a general yield criterion is satisfied. A. E. Green (Providence, R.I.).

Trifan, D. A minimum principle of plasticity. *Quart. Appl. Math.* 13 (1955), 337-339.

In a previous paper [D. Trifan, same *Quart.* 7 (1949), 201-211; MR 10, 760] a proof was given for a minimum principle of a material obeying the stress-strain relations proposed by the author. This paper is concerned with the removal of a restriction imposed on the admissible strain rates in this proof. E. T. Onat (Ankara).

Shield, R. T. Plastic flow in a converging conical channel. *J. Mech. Phys. Solids* 3 (1955), 246-258.

The paper is concerned with the flow of a plastic-rigid, non-hardening material through a conical die. It is assumed that the channel is long enough to neglect the effect of the end conditions, and that the flow is directed towards the apex of the cone. A further assumption is made that the frictional force between the die and the material is constant. It is then shown that under these assumptions the problem is tractable even for a material having a general yield criterion and the method of solution is outlined using such a criterion. The particular cases of the yield conditions of von Mises and Tresca are then considered in detail. Finally the theory is applied to give an approximate description of wire drawing. E. T. Onat (Ankara).

Sokolovskii, V. V. The stressed state of a plastic mass inside a noncircular cone. *Prikl. Mat. Meh.* 16 (1952), 487-490. (Russian)

This paper relates to plastic stress distributions in a region bounded by a general conical surface. The analysis is based upon the Hencky finite equations. Let r , θ , ϕ be spherical polar co-ordinates; other notation is standard. Attention is confined to the case in which $\tau_{\theta\phi} = 0$ and $\tau_{\phi r}$, $\tau_{r\theta}$ are independent of r . It is next supposed that the displacements are purely radial, i.e. $u_\theta = u_\phi = 0$. The problem is then 'statically determinate', and there are four equations (three equations of equilibrium and one yield condition) for the unknown stresses σ_r , $\sigma_\theta = \sigma_\phi$, $\tau_{\phi r}$, $\tau_{r\theta}$. The general solution of this system of equations proceeds through the adoption of expressions, involving the hydrostatic pressure σ and two arbitrary functions $\psi(\theta, \phi)$, $\chi(\theta, \phi)$, for stress distributions that identically satisfy the yield condition. Substitution of these expressions for the stresses in the equilibrium equations then leads to an elliptic system of three differential equations for σ , ψ , χ . It is now further supposed that slip can be prevented at the conical surface. The integration of the above system of equations is then discussed in general terms but no specific problems are solved. R. T. Shield [see the paper reviewed above] has recently solved a problem of the above type, viz. the flow of a plastic mass within a converging circular cone. H. G. Hopkins.

Hopkins, H. G. On the behaviour of infinitely long rigid-plastic beams under transverse concentrated load. *J. Mech. Phys. Solids* 4 (1955), 38-52.

An infinitely long beam is fully restrained at infinity, and is loaded with a time-varying pressure $p(x, t)$, symmetric with respect to the space origin. The beam is made of a rigid perfectly plastic material, and only bending deformations are considered. Under certain general restrictions on p , it is shown that the beam will deform by means of a fixed-yield hinge at $x=0$, and two symmetric outwards-moving hinges.

A specific solution is given for the case when the load is concentrated at the origin, and either this load or the central transverse velocity is prescribed, up until some time $t=\tau$. At $t=\tau$ the load is removed, but it is shown that the beam will not come to rest in any finite time. The solution is further particularized to the central velocity $v=v_0(t/\tau)^n$, and several features of the solution are presented graphically. If $n=0$, the result previously obtained by Conroy [*J. Appl. Mech.* 19 (1952), 465-470] is recovered.

The general method of attack parallels closely the work of Hopkins and Prager [*Z. Angew. Math. Phys.* 5 (1954), 317-330; MR 16, 648] on circular plates, and hence is somewhat different from previous papers by Lee and Symonds [*J. Appl. Mech.* 19 (1952), 308-314] and Symonds and Leth [*J. Mech. Phys. Solids* 2 (1954), 92-102; MR 15, 1006] on transversely loaded beams. P. G. Hodge.

Craemer, H. Idealplastische isotrope und orthotrope Platten bei Vollaussnutzung aller Elemente. *Ing.-Arch.* 23 (1955), 151-158.

The paper is concerned with the plastic design of plates of variable thickness for minimum weight and is based on K. W. Johansen's yield condition [Brudlinieteorier, Teknisk Forlag, Copenhagen, 1943] for reinforced concrete plates. According to this, a plate element is capable of plastic deformation only when at least one of the principal bending moments reaches a critical absolute value m , which depends on the thickness of the element. The author makes the plausible assumption that minimum weight design requires that both principal bending mo-

ments attain this absolute value ("Vollaussnutzung"). If conditions of loading and support are such that the principal bending moments have equal signs throughout the plate ("gleichsinnige Vollaussnutzung"), the bending moment in any direction has the absolute value m , and the twisting moment vanishes. The equation of equilibrium, in conjunction with the boundary condition $m=0$ at a simply supported edge, then determines the variation of m and hence that of the plate thickness. [A similar analysis based on Tresca's yield condition has recently been given by the reviewer, *De Ingenieur*, 67 (1955), no. 48, O.141-O.142.] If, on the other hand, the principal bending moments have unequal signs ("ungleichsinnige Vollaussnutzung"), the equation of equilibrium does not specify the variation of m uniquely, unless the principal directions are known as for instance in the case of rotational symmetry. Examples concerning circular, annular, and rectangular plates are presented. The examples for rectangular plates with principal bending moments of opposite signs are not convincing, because they involve arbitrary assumptions on the pattern formed by the lines indicating the principal directions. A plate design obtained from such an assumption, while adequate, is not necessarily a minimum weight design. The analysis is extended to orthotropic plates. W. Prager (Providence, R.I.).

Franciosi, Vincenzo. Il procedimento del "limit design" per carico non proporzionale. *Ricerca, Napoli* 5 (1954), no. 3, 23-28.

For expository reasons, the determination of the load factor of a structure by limit analysis is usually discussed as if a common load factor applied to all loads [see, however, H. J. Greenberg and W. Prager, *Transactions ASCE* 117 (1952), 447-484, especially, pp. 481-482]. The present author points out that these methods are readily extended to the case where the load factor applied only to some loads while the intensities of the other loads are fixed. W. Prager (Providence, R.I.).

See also: Vorovič, p. 273; Coburn, p. 297; Hochstrasser, p. 302; Boscher, p. 304.

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory, Circuits

Linfoot, E. H. Information theory and optical images. *J. Opt. Soc. Amer.* 45 (1955), 808-819.

The author shows how the methods of information theory can be applied to the analysis of the optical images. The author divides the working field of an instrument into areas such that the diffraction image of the point of such an area is practically indistinguishable from another. Such an area is called an isoplanatism patch. The intensity distribution in an isoplanatism patch is given by a surface integral. The distribution in the image can be obtained by multiplying this integral by a transmission factor. The methods of Fourier analysis can be applied to this problem (or the work of P. M. Duffieux) and the formula thus obtained are formally equivalent to the formulae of information theory, thus permitting the application of the terms of this theory to optical imagery. Distinction between images and noise are discussed, and mathematical formulation given to the

statistical mean-information content, and other terms of information theory. M. Herzberger (Rochester, N.Y.).

Mandl, Georg. Zur Begründung der Strahlenoptik aus der Maxwellschen Feldtheorie. *Acta Phys. Austriaca* 7 (1953), 365-389.

The purpose of this article is to give domains and conditions of validity of geometrical optics for inhomogeneous and anisotropic media based on Maxwell's equations [see also the two papers reviewed below]. In the first two sections a general discussion of the characteristic (eiconal or polarization) equation is presented, including conditions of discontinuity of the electromagnetic fields across a wave front (characteristic surface). [The material of these sections is mostly known; see R. K. Luneberg, *Propagation of electromagnetic waves*, lecture notes, New York Univ., 1948]. In the next section an analysis of the domain of validity of geometrical optics is given by using the standard procedure (Moll-Debye) of expressing the fields in the form $E_i = a_i(x_j) e^{i(k(x_j) - \omega t)/\lambda_0}$ ($i=1, 2, 3$)

and afterwards introducing them either in the wave or Maxwell equations. If the medium is homogeneous, the author shows that the amplitude $a(x)$ must satisfy the inequalities,

$$\left| \frac{\partial a_i}{\partial x_i} \right| \ll \frac{n^2}{6}, \quad \left| \frac{\partial a^2}{\partial x^2} \right| \ll \frac{n^2}{3},$$

where $|n|$ is the refractive index of the medium. Under these conditions the amplitude satisfies equation

$$(a) \quad \frac{\partial \log a^2}{\partial v} = -2H,$$

v being the normal to the surface of constant amplitude and H the mean curvature of the wave fronts, $S = \text{constant}$. If the amplitudes a_i satisfy the Laplace equation, then a rigorous solution of the phase function is found to be $S = \mu a \text{ grad } \varphi / a \text{ curl } a$, φ being a scalar function. For inhomogeneous isotropic media the Moll-Debye transformation of the fields leads to a first-order linear homogeneous equation in the square of the amplitude $a = (a_i)$, provided the following inequality holds:

$$\lambda_0^2 \left| \mu \text{ curl}_i \left(\frac{1}{\mu} \text{ curl } a \right) / a_i \right| \ll n^2$$

and instead of (a) the amplitude satisfies the equation

$$(b) \quad \frac{\partial}{\partial v} \log (a^2 (\varepsilon/\mu)^{1/2} f) = 0,$$

where f is the cross-sectional area of a tube of rays normal to the surfaces of constant phase. *N. Chako.*

Suchy, Kurt. *Schrittweiser Übergang von der Wellenoptik zur Strahlenoptik in inhomogenen anisotropen absorbierenden Medien. II. Lösung der Gleichungen für Wellennormale und Brechungsindex durch WKB-Näherung. Strahlenoptische Reflexion und Alternation.* Ann. Physik (6) 13 (1953), 178-197.

In a previous paper [Ann. Physik. (6) 11 (1952), 113-130; 12 (1953), 423; MR 14, 821] the author derived the characteristic (polarization) equation based on Maxwell's equations for inhomogeneous and anisotropic media by using the Moll-Debye transformation for the fields. It reduces to a first-order non-linear equation in the characteristic (index of refraction) vector n if the medium is stratified. In this article the polarization equation is solved by the WKB method, i.e. by developing n in a power (asymptotic) series in $(i\hbar_0)^{-1}$, the leading term of which gives the geometrical (pure) optics solution [cf. R. K. Luneberg, Mathematical theory of optics, Brown Univ., 1944; MR 6, 107]. It is found that there exist points or regions in the medium, where the successive approximations of n are unbounded. These are regions of reflections of the waves or transformations of the waves from one type to another. Furthermore, explicit expressions are obtained for the first and second terms of the series (asymptotic) when the direction of the vector n coincides with the wave normal. Finally, the author analyses at length the reflection and other critical points and gives conditions for the validity of the WKB solutions. *N. Chako (New York, N.Y.).*

Suchy, Kurt. *Schrittweiser Übergang von der Wellenoptik zur Strahlenoptik in inhomogenen anisotropen absorbierenden Medien. III. Gruppenfortpflanzung.* Ann. Physik (6) 14 (1954), 412-425.

Using some of the results of previous articles [see the preceding review], the author derives expressions for the

propagation of the group, of the amplitude and of the phase of the rays for inhomogeneous and anisotropic media by applying the method of stationary phase. As a result, the principle of Fermat is extended to absorbing media. The paper concludes with a discussion and an analysis of the direction of propagation of the group and group velocity, and the relationship of the former with the Poynting vectors. *N. Chako (New York, N.Y.).*

Vandakurov, Yu. V. *The equations of electron optics for wide beams, taking chromatic aberrations into account, and their application to the investigation of the motion of particles in axially symmetric fields.* Z. Tehn. Fiz. 25 (1955), 1412-1425. (Russian)

The problem studied concerns the motion of particles with relativistic velocities in arbitrary electric and magnetic fields when the basic trajectory, sometimes known as axis of the system, is plane. The equations of electron optics are given taking into account terms which give rise to aberrations up to the third order inclusive. Methods are investigated for taking into account chromatic aberrations, i.e. the presence of dispersion in the initial velocities of particles emitted by a source of extended dimensions. The equations are applied to the determination of trajectories and the calculation of aberrations in axially symmetric fields when the basic trajectory is circular. *J. E. Rosenthal (Passaic, N.J.).*

Cukkerman, I. I. *On finding a magnetic field focussing electron beams of a given type.* Z. Tehn. Fiz. 25 (1955), 853-860. (Russian)

The problem treated is the determination of conditions to be imposed on a magnetic field in order that the axis of an electron beam subjected to it be in the form of a given curve, the electrostatic field being fixed. An added requirement of the problem is that a real image similar to the object be formed in a given normal plane. The following series of examples are worked out for purely magnetic electron-optical systems: 1. The axis of the beam is assumed to be a spiral. 2. It is assumed to be circular, the object and image planes in this case being at an arbitrary angle. 3. The axis of the beam is assumed to be rectilinear. Suggestions are given for the practical realization of the magnetic fields determined. *J. E. Rosenthal.*

Thomas, Johannes. *Zur Theorie der Elektronenbahnen in einer Elektronenschleuder (Betatron).* Math. Nachr. 13 (1955), 73-128.

The motion of electrons inside a betatron is determined by two different methods, relativistic effects being taken into account. The first method uses time as the independent variable. [In this treatment the author follows J. Picht, Optik 6 (1950), 40-55, 61-97, 133-144.]

A perturbation calculation is carried out using the stable circular orbit as the unperturbed solution and assuming that the perturbation caused by a variation in initial conditions is small. The coefficients in the expansion of the solution in powers of these variations are called orbital coefficients "of the first kind" (and the appropriate order). The first order of Gaussian approximation is discussed with regard to the possibility of first-order focussing. Some basic considerations are given concerning higher-order calculations. The second method, using the azimuth as the independent variable, gives the orbital coefficients "of the second kind". The radius of the stable circular orbit and the Hertz vector are calculated. The possibility is discussed of realizing the electromagnetic

field described by the Hertz vector by appropriate shape of the pole pieces. *J. E. Rosenthal* (Passaic, N.J.).

Caldirola, P. *Sull'equazione del moto dell'elettrone nell'elettrodinamica classica.* Nuovo Cimento (9) 10 (1953), 1747-1752.

Consider the following assumptions: (a) there exists a real constant, τ_0 (fundamental interval of time); (b) corresponding to τ_0 , there is a fundamental interval of distance s_0 , $s_0 = c\tau_0$; (c) as $\tau_0 \rightarrow 0$ the equations of the electron reduce to the classical dynamical equations; (d) the equations must be relativistic invariant. On the basis of the first three, the author obtains Dirac's classical equations of the electron [Proc. Roy. Soc. London. Ser. A. 167 (1938), 148] from the equations of motion when one writes them in difference form involving the fundamental time τ_0 , provided the latter is identified with the expression $4e^2/3mc^3$. To satisfy (d) a new difference equation of motion is derived in terms of the proper time τ and τ_0 , which reduces to the former (non-relativistic) when $v^2/c^2 \rightarrow 0$. If the new equation is written in Dirac's form, the reaction force is given by

$$R_s = m_0 \left[c \frac{du_s(\tau)}{ds} + \frac{u_s(\tau - \tau_0) + u_s(\tau)u_\beta(\tau)u_\beta(\tau - \tau_0)}{\tau_0} \right],$$

where $s = \tau c$ and u_α ($\alpha = 1, 2, 3, 4$) are the components of the velocity vector. No attempt has been made to integrate the new equation, even for simple types of external forces. However, in the non-relativistic case, the author has given explicit expressions for the velocity of the electron for the simplest cases of external forces, including periodic and Lorentz forces with constant magnetic field. *N. Chako* (New York, N.Y.).

Elsässer, Hans. *Lichtstreuung an einem Gemisch von dielektrischen Kugeln.* Z. Astrophys. 34 (1954), 50-67.

The paper deals with the problem of scattering of light by a random mixture of dielectric spheres of different size. It is assumed that all spheres are large compared to wavelength and that they are far apart. Using the Mie-Debye theory of scattering by a sphere and the single scattering approximation, the author calculates the intensity of the scattered light in terms of an assumed size distribution function for the scattering particles. The theory is used to explain the distribution of the zodiacal light, on the assumption that it is due to the scattering of sunlight by dust particles. *J. Shmoys.*

Hoffman, William C. *Scattering of electromagnetic waves from a random surface.* Quart. Appl. Math. 13 (1955), 291-304.

Let the surface of a perfectly conducting medium cover a region D of the (x, y) -plane and be given by $z = Z(x, y)$. The reflection of an incident electromagnetic plane wave can be found by means of the usual Kirchhoff approximation. The components of the reflected field at large distance are then functionals of Z . In this paper Z is considered as a random function; the mean values of the reflected field components, and their covariance function, are computed for both polarizations of the incident field. The resulting rather complicated expressions are specialized for a Gaussian random Z , and for a Gaussian random Z with variance independent of x, y . *N. G. van Kampen.*

Avazašvili, D. Z. *A spatial diffraction problem for electromagnetic waves.* Soobšč. Akad. Nauk Gruz. SSR 14 (1953), 321-328. (Russian)

The author studies, using the method given (for plane

problems) by V. D. Kupradze [Boundary problems of the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950; MR 15, 318; Akad. Nauk Gruz. SSR. Trudy Tbiliss. Mat. Inst. 1 (1937), 115-123] a diffraction problem in three-dimensional space relative to a finite number of closed regular non-intersecting surfaces whose interiors form a nested sequence. *J. B. Diaz* (College Park, Md.).

★Avazashvili, D. Z. *Three dimensional problem of diffraction for electromagnetic oscillations.* Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 10 pp. (mimeographed) \$5.00. Translation of the paper reviewed above.

Ingarden, R. S. *A generalization of the Young-Rubinowicz principle in the theory of diffraction.* Acta Phys. Polon. 14 (1955), 77-91. (Russian summary)

The paper contains a generalization of Rubinowicz' method of decomposition of the wave behind a Kirchhoff screen into the "incident" wave u_i and the "diffracted" wave u_D for the case of an arbitrary "geometrical" wave falling on the screen. The calculations are carried out in approximation of Kirchhoff's scalar theory for an arbitrary shape of the screen. Fock's approximate solution of the wave equation [Z. Eksper. Teoret. Fiz. 20 (1950), 961-978; MR 12, 655] is used for the incident wave. The result for u_D is a line integral containing a series expansion in powers of $1/k$ (k , the wave number) as well as of $1/R$ (R , the distance from the integration point to the observation point) and gives Rubinowicz' formula as a special case. In the "parageometric approximation" a specially simple extension of Rubinowicz' result is obtained, being a series whose coefficients define the aberrations of the incident wave along the diffracting edge. (From the author's summary.) *A. E. Heins* (Pittsburgh, Pa.).

Westpfahl, Konradin. *Zur strengen Theorie der Beugung elektromagnetischer Wellen an ebenen Schirmen.* Z. Physik 140 (1955), 354-373.

The scattering of an electromagnetic wave by a disc of "zero thickness" and perfect conductivity is formulated with the aid of the Hertz vectors. This permits a double Fourier integral representation for the field components which may be solved approximately. *A. E. Heins.*

Pfärrmann, Viktor. *Wellenablösung von einer Kegelantenne.* Arch. Elektr. Übertr. 9 (1955), 98-101.

The author considers a quarter-wave conical dipole antenna and derives a theoretical solution for the near field. The method followed is to break the region into three zones and to match boundary conditions on the zonal boundaries. The technique used in the zone nearest the source point is a form of cylindrical mapping attributed to H. H. Meinke [Z. Angew. Physik 1 (1949), 509-516]. *W. K. Saunders* (Washington, D.C.).

Aymerich, Giuseppe. *Guide d'onda anisotrope con "fili" non perfettamente conduttori.* Boll. Un. Mat. Ital. (3) 10 (1955), 165-171.

"In questo lavoro ci siamo proposti di studiare la propagazione delle onde elettromagnetiche in una guida anisotrope nell'ipotesi che i "fili" del guscio siano leggermente assorbenti. Poiché la corrente elettrica si propaga soltanto nella direzione dei fili, è naturale ammettere che l'essere la conduttività finita non debba influire direttamente sulla componente del campo elettrico normale al

filo, ma soltanto su quella, $E_{||}$, parallela al filo, la quale è lecito supporre sia legata, sulla faccia interna e su quella esterna del guscio, alla componente, $H_{||}$, del campo magnetico secondo la direzione tangenziale alla superficie e normale al filo dalla relazione $E_{||} = v H_{||}$.

Servendoci del teorema di reciprocità abbiamo mostrato come le costanti di propagazione della guida non perfettamente conduttrice ed in particolare il coefficiente di assorbimento possono esprimersi in funzione dei "modi" della guida ideale. Abbiamo esaminato anche il caso di un autovale multiple (caso degenerare) mostrando come tale molteplicità possa sparire o ridursi nella guida imperfetta." (From the author's introduction.)

E. T. Copson (St. Andrews).

Fainberg, Ya. B., and Hiznyak, N. A. Artificially anisotropic media. *Ž. Tehn. Fiz.* 25 (1955), 711-719. (Russian)

The paper deals with propagation of electromagnetic waves in circularly cylindrical waveguides, in which the dielectric constant is constant in any cross-section, but a periodic function in the direction of propagation. Such a structure has very similar guiding properties to a waveguide filled with anisotropic dielectric, having different dielectric properties in the longitudinal and transverse directions. The effective anisotropy of periodically spaced thin dielectric discs is calculated.

J. Shmoy's.

Kompaneec, A. S., and Sayasov, Yu. S. Theory of electromagnetic resonators near in form to conical ones. *Ž. Tehn. Fiz.* 25 (1955), 1124-1131. (Russian)

The paper deals with the calculation of the electromagnetic field in a nearly biconical cavity: one bounded by two sheets of a hyperboloid of revolution and a spheroid. The product ϵ of the wave number k and semi-focal distance f is assumed to be small. The problem is solved by separation of variables in spheroidal coordinates; the resulting eigenvalue problems are solved for ϵ small by a perturbation method. The Q of the cavity is calculated approximately.

J. Shmoy's (Brooklyn, N.Y.).

Fedorov, F. I. On polarization of electromagnetic waves. *Dokl. Akad. Nauk SSSR (N.S.)* 102 (1955), 69-71. (Russian)

General criteria are given for ascertaining the polarization of plane monochromatic electromagnetic waves. The problem of polarization of inhomogeneous waves is considered on the basis of these criteria. A number of general conclusions are given concerning the electric and magnetic vectors of the inhomogeneous waves.

J. E. Rosenthal (Passaic, N.J.).

Stepanov, K. M. On the propagation of a wave front in a dispersive medium. *Dopovidi Akad. Nauk Ukrain. RSR* 1955, 63-66. (Ukrainian. Russian summary)

It is shown by the method of characteristics that the equations of propagation of the wave-front in a dispersive medium are the same as those for free space, if it is assumed that the dielectric and magnetic properties of the medium tend to those of free space as the frequency increases without bound.

J. Shmoy's (Brooklyn, N.Y.).

Schellkunoff, S. A. On representation of electromagnetic fields in cavities in terms of natural modes of oscillation. *J. Appl. Phys.* 26 (1955), 1231-1234.

The author takes as an example of a cavity a section of rectangular waveguide with an open end serving as the

window. With this model he is able to demonstrate that the question of completeness of the short-circuited eigenfields raised by T. Teichmann and E. P. Wigner [*J. Appl. Phys.* 24 (1953), 262-267; MR 14, 823] may be satisfactorily resolved within the framework of short-circuited modes, if it is remembered that the short circuit must be such as to conform to the impressed field. Thus, with the waveguide example, the short circuit must be provided by a grid of wires parallel to the impressed electric field and not by a perfectly conducting plate.

A second approach to the problem is also given. This depends on the use of the continuity of the expression for the magnetic fields from which the electric field is then derived. This must be done without resort to differentiation of the series for the electric field, for this series is only conditionally convergent on the window boundary.

W. K. Saunders (Washington, D.C.).

Ferrari, Italo. Multipoli e onde di Schellkunoff. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) 17 (1954), 32-37.

The purpose of this note is to show that the magnetic field produced by a multipole radiator of order n is equivalent to a linear combination of the first n -transverse magnetic spherical wave modes (TMSW) with coefficient independent of the radial distance r . The procedure is as follows. Starting with the magnetic field H_0 due to a Hertzian radiator with moment along the z -axis, the n th-multipole field, H_n , is given by the n th derivative of H_0 with respect to z . The multipole field is expressed in a finite power series of $\cos \theta = z/r$ (polynomial of degree n in $\cos \theta$), the coefficients of which depend on r only. The Legendre functions entering in the (TMSW) modes are also expressed in polynomial series in $\cos \theta$. By equating the former series of H_n with the series involving the (TMSW) modes a set of $(n+1)$ non-homogeneous equations in the unknown coefficients of expansion of H_n in terms of the modes is obtained. This set possesses a unique solution. Furthermore, the coefficients of expansion are shown to be independent of r . The author describes a method of determining the coefficients which results in a recurrence relation of a fairly simple type. Finally, the coefficients for $n=1, 2$ and 3 , the latter being a octopole field, are given.

N. Chako.

Nardini, Renato. Su particolari onde cilindriche della magneto-idrodinamica. *Boll. Un. Mat. Ital.* (3) 10 (1955), 349-362.

This paper is concerned with the determination of cylindrical waves in a homogeneous electrically-conducting viscous incompressible fluid. The equations governing the motion are:

$$\text{rot } \mathbf{H} = \mathbf{i}, \quad \text{rot } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$

$$\mathbf{i} = \gamma(\mathbf{E} + \mu \mathbf{v} \wedge \mathbf{H}), \quad \text{div } \mathbf{H} = 0,$$

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \text{grad } p + \frac{\mu}{\rho} \mathbf{i} \wedge \mathbf{H} + \text{grad } U - \nu \text{rot rot } \mathbf{v},$$

$$\text{div } \mathbf{v} = 0$$

in the obvious notation. The author tries to solve these in the case when all the variables depend only on the time t and the radial coordinate r in cylindrical coordinates (r, θ, z) . It turns out that, if the radial component v_r of the velocity vanishes, $H_r = \lambda/r$ where λ is a constant.

The general problem splits up into two: in the first, two third-order equations for H_z and v_z are obtained, in the second for H_θ and v_θ , and the solution of these would be very difficult. The author determines what he calls semi-stationary solutions; by this he means in, say, the first case that H_z depends only on r but not on t .

If, however, the fluid is non-viscous ($\nu=0$) and perfectly conducting ($\rho=\infty$), the equation for H_z simplifies to

$$\frac{\partial^2 H_z}{\partial t^2} = \frac{\alpha^2}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial H_z}{\partial r} \right),$$

where $\alpha = \lambda/(\mu/\rho)^{1/2}$, and this has the general solution

$$H_z = G_1 \left(t - \frac{r^2 - r_0^2}{2\alpha} \right) + G_2 \left(t + \frac{r^2 - r_0^2}{2\alpha} \right)$$

and the corresponding formula for v_z is readily found. This solution represents cylindrical waves which expand or contract with velocity $\alpha/r = (\mu/\rho)^{1/2} H_r$. But it does not seem possible to obtain exact formulae for H_θ , v_θ in this case; only approximate formulae valid for large r are found.

E. T. Copson (St. Andrews).

★ **Rose, M. E. Multipole fields.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1955. viii+99 pp. \$4.95.

Multipole fields for a 2^L -pole are those solutions of the Maxwell equations which have the following transformation properties: Under a rotation in Euclidean three-space the solutions are permuted among themselves in accordance with a $(2L+1)$ -dimensional irreducible representation of this rotation group, and under a reflection in the origin of Euclidean three-space the associated non-vanishing magnetic field H transforms as

$$H^* = (-1)^L H \text{ or } H^* = (-1)^{L+1} H.$$

The author gives a concise, well-written and self-contained account of the mathematical formalism required for and the description of the various applications to be considered: mainly, internal conversion of orbital electrons, Auger transitions and emission of gamma radiation. A general knowledge of quantum mechanics and electromagnetic theory is assumed; however, the first chapter gives a brief summary of the Maxwell theory. Chapters II and III give a discussion of mathematical properties of multipole fields used in the applications which are treated in the last three chapters. It is in these chapters that a knowledge of quantum mechanics is presupposed.

A. H. Taub (Urbana, Ill.).

★ **Panofsky, Wolfgang K. H., and Phillips, Melba. Classical electricity and magnetism.** Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1955. xi+400 pp. \$8.50.

The book is written to emphasize those aspects of classical electricity and magnetism which are of the most use to the modern experimental and theoretical physicist. It is designed as a suitable basis for an understanding of the quantum theory of matter and radiation, and the level of presentation is one suitable for a first or second year graduate course. In range of material the book does not follow the usual American or English texts but hews closer to the pattern of Abraham and Becker [Theorie der Elektrizität, Bd. II, 6. Aufl., Teubner, Leipzig, 1933]. The latter chapters in fact make available much of the material contained in the untranslated second volume of the German work.

The first five chapters contain material on the founda-

tions of the potential theory and on the solutions of the potential equation in two- and three-dimensional coordinate systems. The illustrative examples are judiciously chosen to lie between trivial ones and ones of great algebraic complexity. A Green's function solution employing the Dirac delta function in a manner current in periodical literature is given for several problems. The sixth chapter discusses the energy relationships in the electrostatic field and includes material on the stress tensor. The seventh chapter is on currents, and the eighth on magnetic substances and magnetic boundary-value problems. The development of Maxwell's equations in both stationary and moving media is the subject of the ninth chapter. The tenth chapter discusses energy and momentum relationships in the electromagnetic field. The two following chapters derive solutions to the scalar and vector wave equation in free space and in the presence of metal boundaries. Illustrative examples chosen include the answers to problems of scattering from spheres and cylinders, and propagation in waveguides. The thirteenth chapter introduces fields of discontinuous and non-periodic sources. The former is handled by the introduction of a single Hertz vector and the latter by the use of the Fourier transform. Throughout these chapters on Maxwell's equations and the preceding ones, energy and momentum concepts are emphasized as are applications of the material to atomic physics and engineering.

Starting with chapter fourteen there commences a discussion of special relativity. The experimental evidence is presented and followed by a check list on the various theories proposed before 1905. The development is continued in chapter fifteen, where there is a discussion of the Einstein postulates followed by the four Gedankenexperimente which result in the Lorentz transformation and its Minkowski geometry. Chapter sixteen covers relativistic mechanics and seventeen a covariant formulation of electrodynamics. Chapters eighteen and nineteen are concerned with the potentials of uniformly moving and accelerated particles. This material is handled primarily with a classical attack based on the Liénard-Wiechert potentials. Chapters twenty and twenty-one pursue this attack upon problems of radiation reaction, scattering, and dispersion. The authors return to the relativistic formulation in chapter twenty-two and derive and apply the laws of conservation in the electromagnetic field. Chapter twenty-three contains a description of the motion of a charged particle in the four-dimensional world while the final chapter takes as its subject Hamilton's variational formulation of Maxwell's equations.

The compression of such a wealth of material, plus two appendices and an index, into but 400 pages has resulted inevitably in a somewhat uneven presentation. Some sections, such as the one introducing special relativity are reasonably full and lucid, while others, such as the paragraphs on the conceptually difficult delta function, are exceedingly brief. Viewed from the standpoint of the originally devised lecture notes, this unevenness is not particularly important, but it does seriously militate against the book as one for independent study. Moreover, the transition from the lecture notes to the published volume, has not been completed in a very satisfactory manner. The writing is sometimes awkward, and on occasion says what the authors evidently do not intend. A particularly unfortunate example is to be found on page 202 where a badly written paragraph is followed by the statement that $\vec{r} \times \nabla \psi$ is a solution to the scalar wave

equation. The publishers, although having done an excellent job in binding and typesetting, appear to have slighted their responsibility for editorial guidance. This is unfortunate as the book is potentially a very useful one having a scope of material, and an approach to this material, for which there is considerable need.

W. K. Saunders (Washington, D.C.).

Moon, Parry, and Spencer, Domina Eberle. Some electromagnetic paradoxes. *J. Franklin Inst.* 260 (1955), 373-395.

Durand, Emile. Théorie générale des masses magnétiques au repos et en mouvement. *Rev. Gén. Elec.* 64 (1955), 350-356.

A general theory of magnetic masses at rest and in motion is developed in close analogy to that of electric charges. It is shown that there is a system of fictitious magnetic masses at rest equivalent to the system of real electric charges and that calculations based on the fictitious system may be easier to carry out than those based on the real system. It is also shown that fictitious magnetic currents produced by the motion of magnetic masses are equivalent to real electric systems. This equivalence is of special interest in the study of electrically polarized media. The electric field produced by the motion of permanent magnets is studied in the following two cases: 1) where the magnet is cylindrical of indefinite length and moves parallel to the direction of its generating lines; 2) where the magnet is axially symmetrical and rotates around its axis. A number of formulas are derived pertaining to moving magnetic masses, which permit the calculation of the electric field and the electric induction produced by a moving magnet. The case of a multiplicity of electric systems is also discussed, and it is shown that for purposes of calculation it may be convenient to replace some of them by equivalent fictitious magnetic systems.

J. E. Rosenthal (Passaic, N.J.).

Durand, Emile. Une nouvelle transformation vectorielle d'intégrale curviligne en intégrale de surface et son application à la magnéto-statique. *C. R. Acad. Sci. Paris* 241 (1955), 594-596.

Using tensor analysis and assuming that the scalar ϕ is a component of an antisymmetric tensor or of a vector, the basic formula

$$\int_C \phi d\mathbf{l} = \iint_S [\mathbf{n} \times \text{grad } \phi] dS$$

is transformed into

$$\int_C [d\mathbf{l} \times \mathbf{A}] = \iint_S \{(\mathbf{n} \cdot \text{grad}) \mathbf{A} + [\mathbf{n} \times \text{rot } \mathbf{A}] - \mathbf{n} \cdot \text{div } \mathbf{A}\} dS.$$

In these formulas the vector $d\mathbf{l}$ is an element of an oriented curve C , \mathbf{n} is a unit vector normal to the surface S bounded by the curve C . In the applications, the vector \mathbf{A} is taken to represent (1) an induction \mathbf{B} acting on a current; (2) the vector $\text{grad}'(1/r)$, the prime being used to denote the coordinates with respect to which the operation is performed. J. E. Rosenthal (Passaic, N.J.).

***Libois, P., Géhéniau, J., et Debever, R.** Espaces de l'électromagnétisme. IIIe Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 36-39. Fédération belge des Sociétés Scientifiques, Bruxelles.

In electrostatics, or in magnetostatics, intensity furnishes a covariant vector (gradient of potential) and

induction a contravariant vector. General electromagnetic theory, viewed in an amorphous locally affine 4-space, furnishes two skew-symmetric tensors, one covariant (H_{ab}) and the other contravariant (K^{ab}). The authors describe briefly the geometrical implications of the equations $H_{ab}dS^{ab}=0$, $K^{ab}d\Sigma_{ab}=0$, where dS^{ab} are the components of an element of area and $d\Sigma_{ab}=dS^{14}$, etc. [For more detailed work on electromagnetic spaces, see Synge, *Proc. Symposia Appl. Math.* 2 (1950), 21-48; MR 11, 401; Géhéniau, *Acad. Roy. Belg. Bull. Cl. Sci.* (5) 37 (1951), 324-332; MR 13, 279; Debever, *Colloque de géométrie différentielle*, Louvain, 1951, Thone, Liège, 1951, pp. 217-233; MR 13, 580; Debever and Géhéniau, *Acad. Roy. Belg. Bull. Cl. Sci.* (5) 41 (1955), 346-355.]

J. L. Synge (Dublin).

Zaicev, G. A. Application of real spinors to the description of the electromagnetic field. *Ž. Eksper. Teoret. Fiz.* 25 (1953), 675-678. (Russian)

If the electromagnetic field $F^{ab}=(E, H)$ satisfies at every point of space-time the conditions

$$E^2 - H^2 = E \cdot H = 0,$$

then F^{ab} determines a real spinor ψ in the sense of a previous paper [same vol., 667-674; MR 17, 330]. In particular, these conditions hold for a plane electromagnetic wave. It can then be proved that the Maxwell equations for (E, H) are equivalent to an equation for ψ which has the form of a Dirac equation with rest-mass zero.

F. J. Dyson (Princeton, N.J.).

Zaicev, G. A. Description of an electromagnetic field by means of matrices. *Ž. Eksper. Teoret. Fiz.* 28 (1955), 524-529. (Russian)

It is shown that the Maxwell equations can be written and manipulated in a very simple way by associating with the field-tensor F^{ab} a matrix $F = \frac{1}{2} F^{ab} R_a R_b$. The R_a are special numerical matrices with commutation laws chosen so as to make the formalism manifestly covariant. They are defined in an earlier paper by the author [same Ž. 25 (1953), 667-674; MR 17, 330].

F. J. Dyson.

Bullard, Edward, and Gellman, H. Homogeneous dynamos and terrestrial magnetism. *Philos. Trans. Roy. London. Ser. A.* 247 (1954), 213-278.

The mathematical problem underlying the construction of a homogeneous dynamo is the following. A divergence-free vector \mathbf{H} satisfies the equation

$$(1) \quad \text{curl} [\text{curl } \mathbf{H}] = V \text{curl} (\mathbf{v} \times \mathbf{H})$$

where \mathbf{v} is some specified solenoidal vector and V is a characteristic value parameter to be determined. Equation (1) is satisfied inside a sphere of unit radius; outside this sphere \mathbf{H} is of the form

$$(2) \quad \mathbf{H} = \sum_{m,n} \text{grad}(Y_n^m/r^{n+1}) \quad (|r| > 1)$$

where Y_n^m is a spherical harmonic of order n . The boundary conditions on \mathbf{H} are that \mathbf{H} be continuous on $|r|=1$ and further that the normal component of $\text{curl } \mathbf{H}$ vanishes on this surface. The present paper is an attempt to show that for certain special forms of \mathbf{v} , the problem allows a real characteristic value for V .

The method consists in expressing both \mathbf{v} and \mathbf{H} in terms of toroidal (T_n^m) and poloidal (S_n^m) fields in the manner made familiar by Elsasser [*Phys. Rev.* (2) 69

(1946), 106-116; MR 7, 401]:

$$\begin{aligned} (T_n^m)_r &= 0 & ; & \quad (S_n^m)_r = -\frac{n(n+1)}{r^2} S_n^m(r) Y_n^m \\ (T_n^m)_\theta &= \frac{T_n^m(r)}{r \sin \theta} \frac{\partial Y_n^m}{\partial \varphi} & ; & \quad (S_n^m)_\theta = \frac{1}{r} \frac{\partial S_n^m(r)}{\partial r} \frac{\partial Y_n^m}{\partial \theta} \\ (T_n^m)_\varphi &= -\frac{T_n^m(r)}{r} \frac{\partial Y_n^m}{\partial \theta} & ; & \quad (S_n^m)_\varphi = \frac{1}{r \sin \theta} \frac{\partial S_n^m(r)}{\partial r} \frac{\partial Y_n^m}{\partial \varphi} \end{aligned}$$

where $T_n^m(r)$ and $S_n^m(r)$ are radial functions to be determined.

The form for \mathbf{v} chosen is a superposition of a toroidal field of the type T_1^0 (with a scalar $T=r^3(1-r^2)$) and a poloidal field, S_2^2 :

$$\mathbf{v} = T_1^0 + \varepsilon S_2^2$$

where ε is a constant. The magnetic field \mathbf{H} is expanded in terms of the same toroidal and poloidal functions. In this manner a simultaneous system of equations for the radial functions $T_n^m(r)$ and $S_n^m(r)$ is obtained. In order to solve this system approximately the authors break off the spherical harmonic series after a finite number of terms and solve the resulting equations (replaced by an equivalent finite matrix system) by making use of the electronic computing machine of the (British) National Physical Laboratory. In their work they have included all spherical harmonics of order up to and including four. Specifically, the modes included are $S_1, S_2, S_2^{2\theta}, S_2^{2\varphi}, T_2, T_2^{2\theta}, T_2^{2\varphi}, T_4, T_4^{2\theta}, T_4^{2\varphi}, T_4^{4\theta}$ and $T_4^{4\varphi}$ (where c and s distinguish spherical harmonic with $\cos m\varphi$ and $\sin m\varphi$, respectively). For $\varepsilon=5$ and 100 the values of V determined are 83.90 and 67.413, respectively.

The convergence of the procedure used for determining V is discussed. But, as the authors admit, the evidence provided by the numerical work is inconclusive.

S. Chandrasekhar (Williams Bay, Wis.).

Duffin, R. J. Elementary operations which generate network matrices. Proc. Amer. Math. Soc. 6 (1955), 335-339.

The positive real matrix functions which occur in the synthesis of n -terminal pair networks is shown to consist precisely of the matrices generated from the identity matrix I and zI by simple analogues of the operations of the corresponding theorem for positive real functions.

C. Saltzer (Cleveland, Ohio).

Tasny-Tschiassny, L. Asymmetrical finite difference network for tensor conductivities. Quart. Appl. Math. 12 (1955), 417-420.

The method of Macneal [same Quart. 11 (1953), 295-310; MR 15, 257] for determining an analogue network to approximate the solution of the boundary-value problem in two dimensions for the electrical potential in an isotropically conducting material is generalized to the case of an anisotropic medium.

C. Saltzer.

Okada, S. Topology applied to switching circuits. Proceedings of the symposium on information networks, New York, April, 1954, pp. 267-290. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

By means of incidence matrices an analogue of the mesh method of circuit theory is used to determine the Boolean functions of contact networks and the synthesis problem is discussed.

C. Saltzer (Cleveland, Ohio).

Doyle, Thomas C. Topological and dynamical invariant theory of an electrical network. J. Math. Phys. 34 (1955), 81-94.

The notion of a subgraph is generalized by the use of real numbers as coefficients and an invariant formulation of the Helmholtz-Maxwell mesh method is given. The fundamental theorem of lumped-constant parameter networks is shown to be equivalent to a reduction of the representation of a network to canonical basis.

C. Saltzer (Cleveland, Ohio).

Nedelcu, Mariana. Analyse de certains schémas à relais temporisés. Acad. R. P. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. 7 (1955), 19-32. (Romanian. Russian and French summaries)

Relay systems whose elements do not respond instantly are analysed by the algebra of logic. From the wiring diagram one sets up recursive formulas whose variables are elements of a Boolean algebra and whose solution determines the performance of the system. A number of simple examples are given.

D. H. Lehmer.

Moisil, Gr. C. Théorie algébrique du fonctionnement des schémas à relais à contacts échelonnés. Acad. R. P. Romîne. Stud. Cerc. Mat. 6 (1955), 7-53. (Romanian. Russian and French summaries)

This is an elaborate account from first principles of an algebraic treatment of systems of relays and other contact mechanisms using Boolean algebra. Both analysis and synthesis of such systems are treated by recursive functions whose many variables take on the values 0 and 1. Eleven examples with periodic solutions are worked out in great detail.

D. H. Lehmer (Berkeley, Calif.).

Moisil, Gr. C., et Ioanin, Gh. Sur le fonctionnement des schémas à boutons réels. Acad. R. P. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. 7 (1955), 33-49. (Romanian. Russian and French summaries)

The analysis of the paper reviewed above is applied to push buttons, simple and compound, shorting and non-shortening, which do not operate instantly.

D. H. Lehmer (Berkeley, Calif.).

Povarov, G. M. A new method of synthesis of symmetric contact schemes. Dopovidi Akad. Nauk Ukrain. RSR 1955, 115-117. (Ukrainian. Russian summary)

The author gives a very brief account of a use of Boolean algebra as applied to relay systems. The method stresses the use of the elementary symmetric functions of the variables involved. The concepts are those developed in the textbook of Gavrilov [Theory of relay-contact schemes, Izdat. Akad. Nauk SSR, Moscow-Leningrad, 1950; MR 12, 225].

D. H. Lehmer (Berkeley, Calif.).

Quantum Mechanics

Born, Max. L'interprétation statistique de la mécanique quantique. (Conférence Nobel, 1954.) J. Phys. Radium (B) 16 (1955), 737-743.

de Broglie, Louis. Ondes régulières et ondes à région singulière en Mécanique ondulatoire. C. R. Acad. Sci. Paris 241 (1955), 345-348.

A general proof is outlined that for each regular solution of the Schrödinger wave equation there exists a

second solution with a moving point singularity which follows one of the lines of flow of the regular solution. The proof applies also to the Dirac and Klein-Gordon equations. It is suggested that this may permit the notion of localized particle motion in wave mechanics, and offer an alternative to the purely statistical interpretation.

P. T. Matthews (Birmingham).

Schönberg, M. A non-linear generalization of the Schrödinger and Dirac equations. II. Nuovo Cimento (9) 12 (1954), 649-667.

A generalization proposed earlier [Nuovo Cimento (9) 11 (1954), 674-682; MR 17, 219] for the Hamilton-Jacobi, Schrödinger and Dirac equations is here applied to all theories of fields, waves or continuous media which are derived from a Lagrange principle and admit gauge invariance of the first kind. For complex fields the generalization consists in adding to the derivatives of the field function U expressions of the forms $(i/\hbar)\lambda(\partial\mu/\partial x_i)U$, where λ and μ are functions of space and time. It is shown that these additional terms can be regarded as electromagnetic potentials giving rise to vanishing Lorentz forces for the possible motions of the system. In the case of hydrodynamics λ and μ correspond to Clebsch parameters allowing for rotational motion. On the basis of this fact theorems similar to the vortex theorems of Helmholtz are derived for the general case. Mention is made of the possibility of expressing electromagnetic potentials by means of Clebsch parameters.

L. Van Hove.

Wightman, A. S., and Schweber, S. S. Configuration space methods in relativistic quantum field theory. I. Phys. Rev. (2) 98 (1955), 812-837.

The paper is composed of three almost independent parts. Part 1. The following problem is posed. To find all sets of linear operators a_j ($j=1, 2, \dots$), everywhere defined in a Hilbert space \mathfrak{H} , and satisfying

$$(1) \quad a_j a_k + a_k a_j = 0, \quad a_j a_k^* + a_k^* a_j = \delta_{jk} \quad (j, k=1, 2, \dots).$$

The relations (1) are the canonical anticommutation rules for the particle operators of a field quantized according to Fermi statistics. Therefore the problem of enumerating the representations of the algebra (1) is equivalent to enumerating the possible types of "particle description" of a quantum field. The problem is handled with full mathematical rigor, although the details of proofs are not given here but are to be published in a forthcoming paper by L. Gårding and A. S. Wightman. A complete solution is not reached but the following substantial results are obtained.

The operator $N_j = a_j^* a_j$ has for each j the eigenvalues 0 and 1. If α is any infinite sequence $(\alpha_1, \alpha_2, \dots)$ of zeros and ones, an operator E_α is defined by

$$(2) \quad E_\alpha = \prod_j N_j^{\alpha_j} (1 - N_j)^{1 - \alpha_j}.$$

A representation of (1) is called continuous if all E_α are zero. It is called discrete if there is no non-trivial subspace on which all E_α vanish. Theorem: Every representation of (1) is a direct sum of a discrete and a continuous representation. The proof is elementary. It is shown first that $E = \sum_\alpha E_\alpha$ is a projection operator, next that a representation in \mathfrak{H} is a direct sum of two representations in the subspaces $E\mathfrak{H}$ and $(1-E)\mathfrak{H}$, and finally that these representations are discrete and continuous respectively.

The problem of discrete representations is completely

solved as follows. Theorem: Every discrete representation of (1) by bounded operators is a direct sum of irreducible representations. There is precisely one irreducible discrete representation R_α corresponding to each α , and R_α and R_β are unitarily equivalent if and only if the sequences α, β differ in only a finite number of places, in which case we write $\alpha = \beta$. The proof is again elementary and is based on the fact that the manifolds $M_\alpha = \sum_{(\beta=\alpha)} E_\beta \mathfrak{H}$ are each invariant under all operations of the algebra; the original representation is split into a sum of representations in the various M_α , and the representation in M_α is the sum of a number of repetitions of R_α . The explicit construction of R_α is as follows. A point of Hilbert space is a complex-valued function $f(\beta)$ defined for each $\beta = \alpha$ and satisfying $\sum_\beta |f(\beta)|^2 < \infty$. The scalar product is $(f, g) = \sum_\beta f(\beta) \bar{g}(\beta)$. The operators a_j and a_j^* are defined by

$$(3) \quad \begin{aligned} (a_j f)(\beta) &= (1 - \beta_j) (-1)^{B_j} f(T_j \beta), \\ (a_j^* f)(\beta) &= \beta_j (-1)^{B_j} f(T_j \beta), \\ B_j &= \sum_{k=1}^{j-1} \beta_k, \end{aligned}$$

where $(T_j \beta)$ is obtained from β by changing β_j into $(1 - \beta_j)$. In the special case $\alpha = 0$, this representation reduces to the one customarily used in quantum field theory [P. Jordan and E. P. Wigner, Z. Physik 47 (1928), 631-651, p. 650].

The analysis of continuous representations is much more complicated because in this case a representation need not be a discrete sum of irreducible representations. An explicit form for the general representation is exhibited; it is a direct generalization of (3). Necessary and sufficient conditions for two such representations to be equivalent are obtained; these conditions are complicated and will not be stated here. Three examples of different types of continuous representation are displayed. The first and simplest is as follows. Points of Hilbert space are complex-valued functions $f(\beta)$, where β is identified with the binary digital expansion of a real number in the interval $[0, 1]$. The scalar product is $\int_0^1 f(\beta) \bar{g}(\beta) d\beta$. The operators are

$$(4) \quad \begin{aligned} a_j f(\beta) &= (1 - \beta_j) (-1)^{B_j} \theta_j f(T_j \beta), \\ a_j^* f(\beta) &= \beta_j (-1)^{B_j} \theta_j f(T_j \beta), \\ \theta_j &= \exp(\pi i \beta_j), \end{aligned}$$

where the θ_j are a fixed sequence of numbers with $|\theta_j| = 1$. A necessary and sufficient condition for equivalence of two representations (4) is

$$(5) \quad \sum_1^\infty |\theta_k - \theta'_k|^2 < \infty.$$

The second and third examples are more sophisticated and constitute rings of operators of classes II₁ and III_∞ [J. von Neumann, Ann. of Math. (2) 41 (1940), 94-161; MR 1, 146].

A parallel analysis is made of the representations of the algebra obtained by changing the sign in (1) from plus to minus; these correspond to the particle description of a Bose field.

Part 2. Explicit formulae are derived for the operators representing position, momentum and spin of individual particles in quantum field theory. The results codify and in some cases correct earlier derivations.

Part 3. A general formalism is constructed to describe the ways in which wave-functions and field-operators may transform under operations of the Lorentz group.

In particular, simple definitions are included for the operations of time-inversion and charge-conjugation.

F. J. Dyson (Princeton, N.J.).

Zalcev, G. A. Concrete representation of states of particles with spin $\frac{1}{2}$ in nonrelativistic quantum mechanics. *Z. Eksper. Teoret. Fiz.* 25 (1953), 653-666. (Russian)

In non-relativistic quantum mechanics the state of a particle of spin $\frac{1}{2}$ is defined by a wave-function $\xi(r)$ which is a spinor with two complex components ξ_1, ξ_2 . This paper describes a direct geometrical interpretation of ξ .

Write $\xi_1 = \psi_1 + i\psi_2, \xi_2 = \psi_3 + i\psi_4$, so that the wave-function becomes a column-matrix ψ with 4 real components. The probability density is

$$\rho = \rho(r) = (\xi^* \xi) = (\psi \psi).$$

In the ψ -representation the Pauli spin-matrices become

$$S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The imaginary unit i becomes

$$I = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

A fifth matrix is defined by

$$B = \begin{bmatrix} 0 & \cos u & 0 & -\sin u \\ -\cos u & 0 & \sin u & 0 \\ 0 & -\sin u & 0 & -\cos u \\ \sin u & 0 & \cos u & 0 \end{bmatrix},$$

where u is an arbitrary fixed angle. Three vectors (a, b, c) are defined by

$$a_k = (\psi B I S_k \psi), \quad b_k = (\psi B S_k \psi), \quad c_k = (\psi S_k \psi).$$

Then it is easy to prove: (i) the vectors (a, b, c) form an orthogonal triad, (ii) each vector has length ρ , (iii) each transforms as a vector when the spinor wave-function ξ is transformed by a rotation of the coordinate axes, and (iv) when (a, b, c) are fixed the wave-function ψ is determined up to a sign ± 1 .

The field of vector triads (a, b, c) therefore gives a complete representation of the state of the particle. This representation has the advantage of being single-valued, unlike the spinor ξ which is double-valued. The arbitrariness in the choice of the angle u means that the absolute orientation of the vectors (a, b) has no physical significance. A change from u to $(u+v)$ produces a rotation of (a, b) through an angle v about the direction c at every point of space. Such a rotation is equivalent to multiplying the spinor wave-function ξ by an unobservable constant phase-factor $\exp[\frac{1}{2}iv]$. F. J. Dyson.

Zalcev, G. A. Real spinors in four-dimensional Minkowski space. *Z. Eksper. Teoret. Fiz.* 25 (1953), 667-674. (Russian)

The algebra of real spinors developed in the paper reviewed above is here extended to 4-dimensional space-time. Essentially no change is required for the extension; in particular, the spinors remain 4-component quantities and do not (like Dirac spinors) involve 8 real components. In relativistic notation, the main result of the preceding paper may be stated as follows: every real spinor ψ

determines an antisymmetric tensor $F^{\alpha\beta}$ satisfying the two identities

$$F^{\alpha\beta} F^{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} = 0.$$

Conversely, every such $F^{\alpha\beta}$ determines ψ up to a sign ± 1 . F. J. Dyson (Princeton, N.J.).

Zalcev, G. A. Relativistically invariant equations for an electron which replace Dirac's system of equations. *Z. Eksper. Teoret. Fiz.* 28 (1955), 530-540. (Russian)

Using the spinor algebra developed in a previous paper [see preceding review] the following is proposed as a relativistically invariant wave-equation for an electron:

$$[-[\hbar J(\partial/\partial x_4) + (e/c)A_4]^2 + \sum_k [-\hbar J(\partial/\partial x_k) + (e/c)A_k]^2 + m_0^2 c^4 - (e\hbar/c)JF]\psi = 0.$$

Formally this is almost but not quite identical with the iterated Dirac equation. The basic difference between the two equations is that the author's ψ is real whereas Dirac's ψ is complex. The author solves his equation explicitly for the bound states in a Coulomb field and shows that the energy-levels are the same as in Dirac's theory. He does not exhibit any consequence of his equation which differs from Dirac's theory.

The reviewer is unable to understand how any states involving non-zero average current can be constructed with real spinors. The author does not discuss this question. The bound states which he constructs have, of course, no average current. F. J. Dyson.

Zalcev, G. A. Tensors characterized by two real spinors. *Z. Eksper. Teoret. Fiz.* 29 (1955), 166-175. (Russian)

A detailed and formal algebraic discussion of the relation between two real spinors ψ_1, ψ_2 and their associated vector-triads, defined in the author's earlier paper reviewed third above. The main result is that, when the rotation from one triad to the other is characterized by the Euler angles θ, φ, ψ , then the transformation between the two spinors is $\psi_2 = K\psi_1$, where K is a quaternion with the components

$$k_1 = \tau \sin \frac{1}{2}\theta \cos \frac{1}{2}(\psi - \varphi), \quad k_2 = \tau \sin \frac{1}{2}\theta \sin \frac{1}{2}(\psi - \varphi), \\ k_3 = \tau \cos \frac{1}{2}\theta \sin \frac{1}{2}(\psi - \varphi), \quad k_4 = -\tau \cos \frac{1}{2}\theta \cos \frac{1}{2}(\psi - \varphi).$$

F. J. Dyson (Princeton, N.J.).

Zalcev, G. A. On the interpretation of Dirac's equations for an electron. *Z. Eksper. Teoret. Fiz.* 29 (1955), 176-180. (Russian)

The author uses his calculus of real spinors [see the paper reviewed above] to reinterpret the Dirac equation. In this formalism the Dirac equation splits into two equations for two real spinors, which are uncoupled for a free electron but coupled when there is an electromagnetic field.

F. J. Dyson (Princeton, N.J.).

Heisenberg, W., Kortel, F., und Mitter, H. Zur Quantentheorie nichtlinearer Wellengleichungen. III. *Z. Naturf.* 10a (1955), 425-446.

The exploration of a model for a theory of elementary particles, begun in two earlier papers [W. Heisenberg, *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa* 153 (1953), 111-127; *Z. Naturf.* 9a (1954), 292-303; *MR* 15, 914, 915], is here carried through in detail. The quantitative results which are obtained from the model are the following. (i) There exists a state representing a Fermi particle with spin $\frac{1}{2}$ and mass $7.43 \cdot 10^{-1}$, where l is

the "fundamental length" of the model. This confirms the rough calculation in the second paper cited above, which gave a mass $7.45 l^{-1}$. (ii) There exists a state representing a Bose particle with spin 1 and mass zero, having two possible polarization-states and behaving in all purely kinematical respects like a photon. (iii) There exists a long-range force between two Fermi particles, derived from a potential

$$V = (\sigma_1 \cdot \sigma_2) (g^2/r),$$

thus behaving like a spin-dependent Coulomb force. (iv) The "effective charge" g appearing in the long-range potential is a computable quantity, and the "effective fine-structure constant" has the value $(g^2/\hbar c) = 0.068$. (v) There exist four states representing Bose particles, one having spin 1 and mass $0.33 l^{-1}$, three having spin 0 and masses $0.95 l^{-1}$, $1.74 l^{-1}$, $3.32 l^{-1}$.

All these results are derived by approximate solution of the equations of the model. Some rather heavy numerical work was required. The qualitative similarity between the results and the observed types of elementary particle is certainly impressive. In this paper nothing is added which substantially changes the logical and theoretical basis of the first author's procedure, which was discussed in the earlier papers cited above.

F. J. Dyson. (Princeton, N.J.).

Sokolik, G. A. On the theory of nonlinear relativistically invariant equations. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 817-820. (Russian)

A formal method is developed by means of which a non-linear partial differential equation for a function φ is translated into a linear functional derivative equation for a functional Ψ . The possible types of linear functional equations can be classified by the known theory of linear representations of the Lorentz group. In this way a classification is obtained for possible types of non-linear field equations governing the motion of a Bose field. Only an outline of the method is presented, without concrete examples.

F. J. Dyson (Princeton, N.J.).

Finkelstein, R. J. On non-local form factors. Nuovo Cimento (10) 1 (1955), 1113-1119.

The author proposes that form factors occurring in non-local theories be derived from a classical lagrangian principle. Methods for dealing with the quantum-mechanical use of these are then outlined.

A. H. Taub.

Pease, Robert L., and Pease, Jane. Necessary condition for positive definite energy. Phys. Rev. (2) 99 (1955), 1600-1601.

A necessary condition for positive definite energy density of the plane wave components, and hence the total energy, of a system described by a first-order wave equation is expressed as a condition on the wave-equation matrices alone.

P. T. Matthews (Birmingham).

Rzewuski, Jan. On the interaction of particles in Feynman's theory. Studia Soc. Sci. Torun. Sect. A. 3 (1954), 1-13. (Polish summary)

This is an expository account of the action-at-a-distance formulation of quantum electrodynamics due to R. P. Feynman [Phys. Rev. (2) 76 (1949), 769-789; MR 11, 765]. The corresponding formalism is also derived for the theory of nucleons interacting through a charged vector meson field.

F. J. Dyson (Princeton, N.J.).

Burton, W. K., and De Borde, A. H. The evaluation of transformation functions by means of the Feynman path integral. Nuovo Cimento (10) 2 (1955), 197-202.

Davison [Proc. Roy. Soc. London. Ser. A. 225 (1954), 252-263; MR 16, 319] has given a simple method for computing the Feynman history integral that gives the transition amplitude of a quantum-mechanical system from a configuration at one time to a second configuration at another time. In this paper, the method is applied to compute the transition amplitude for a free particle, a two-dimensional rigid rotator, and a harmonic oscillator. Two different treatments of the harmonic oscillator are given, with Lagrangians respectively quadratic and linear in the velocities.

A. S. Wightman (Princeton, N.J.).

★ **González Domínguez, Alberto.** On some divergent integrals of quantum electrodynamics. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 53-60. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

A discussion of some divergent integrals which occur in the perturbation-theory treatment of quantum electrodynamics. It is shown that these integrals can be systematically made finite by interpreting them as integrals over distributions in the sense of L. Schwartz [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR 12, 31, 833]. The author is unaware of the parallel work by W. Güttinger [Phys. Rev. (2) 89 (1953), 1004-1119; MR 15, 85].

F. J. Dyson (Princeton, N.J.).

Bocchieri, P., e Loinger, A. Sulla relazione fra la teoria di Tomonaga-Schwinger e quella di Dirac-Fock-Podolsky. Nuovo Cimento (10) 2 (1955), 314-319.

A short, simple, expository account of the equivalence between the many-time formalism of quantum electrodynamics according to Dirac, Fock, and Podolsky [see, for example, P. A. M. Dirac, The principles of quantum mechanics, 2d ed., Oxford, 1935, pp. 286-291], and the non-charge-symmetrical form of Tomonaga-Schwinger field theory. [For another proof of the equivalence, see M. Günther, Phys. Rev. (2) 88 (1952), 1411-1421; MR 14, 707.]

A. S. Wightman (Princeton, N.J.).

Deprit, A. Temperate distributions associated with the Klein-Gordon equation. Nuovo Cimento (9) 12 (1954), 335-350.

The various singular functions encountered in field theory as elementary solutions and Green functions of the Klein-Gordon equation are defined and studied in terms of L. Schwartz's distribution analysis.

L. Van Hove.

Rayski, Jerzy. Simple examples of failure of the standard perturbation methods. Studia Soc. Sci. Torun. Sect. A. 3 (1954), 73-85. (Polish summary)

Example 1. A Klein-Gordon wave-function $\psi(x)$ satisfying the equation

$$(\square - m^2 - \lambda^2)\psi = 0$$

is expanded as a power-series in λ . The exact solutions are of course well-known. The expansion is meaningful and gives the correct solution if the initial values of ψ are given at a finite time t . But if one tries to pass to the limit $t \rightarrow -\infty$ (as is customarily done in quantum field theory) the expansion becomes mathematically ambiguous (as a result of diverging integrals) and an exact solution does not exist.

Example 2. Two fields ψ_1, ψ_2 satisfying

$$(\square - m^2)\psi_{1,2} = \lambda^2\psi_{2,1}.$$

The behavior is precisely the same as in Example 1.

Example 3. A Dirac field ψ in a constant electromagnetic potential A_μ . There again exists no exact solution in which an "unperturbed" initial state of ψ is specified at $t = -\infty$. In this case the mathematical ambiguity in the perturbation expansion is identical with the well-known "photon self-energy" divergence.

F. J. Dyson.

Rayski, Jerzy. Remarks on gauge invariance. *Studia Soc. Sci. Torun. Sect. A.* 3 (1954), 86-91. (Polish summary)

A verbal discussion without mathematical details. It is pointed out that the "gauge invariance of the first kind", i.e. invariance under changes of phase of a complex field by a constant independent of position, is an invariance property of the same kind as invariance under Lorentz transformations. Both invariance-groups are specified by a discrete set of parameters, and both are associated with integral conservation laws (charge and energy-momentum respectively). On the other hand the "gauge-invariance of the second kind", i.e. invariance under a change in phase which is an arbitrary function in space-time, is a quite different kind of invariance, being associated with a group of infinite dimensionality. The author proposes to abandon the requirement of gauge invariance of the second kind, and to look for a new theory of elementary particles which possesses at most an invariance under changes in phase which are linear functions of position (corresponding to constant additions to the electromagnetic potentials).

F. J. Dyson (Princeton, N.J.).

Valatin, J. G. State vector and quantization in an over-all space-time view. *Proc. Roy. Soc. London. Ser. A.* 229 (1955), 221-234.

This paper is mainly a review article surveying a number of recent contributions to the development of quantum field theory. The approach is purely formal, and emphasizes the use of functional methods. The chief novelty is the author's proposal to define the concept of "state" directly in terms of functionals of an external source-function.

F. J. Dyson (Princeton, N.J.).

Halatnikov, I. M. Representation of Green's function in quantum electrodynamics in the form of continuous integrals. *Ž. Eksper. Teoret. Fiz.* 28 (1955), 633-636. (Russian)

The explicit solution of the Green's function equations of quantum electrodynamics is constructed by means of functional integrals. The author takes some care to define the integrals over the Fermi field functions in such a way that the integration variables are c -numbers. Thus he avoids the dubious notion of "anti-commuting c -numbers" which mars other presentations of the functional method.

F. J. Dyson (Princeton, N.J.).

Güttlinger, Werner. Products of improper operators and the renormalization problem of quantum field theory. *Progr. Theoret. Phys.* 13 (1955), 612-626.

This paper is one of a series [*Phys. Rev.* (2) 89 (1953), 1004-1019; *Z. Naturf.* 10a (1955), 257-266; MR 15, 85; 17, 221] in which the possibility of using distribution theory in quantum field theory is explored. The first part of the paper is devoted to a heuristic discussion of the

problem of defining a product distribution for two general distributions. The second part is a summary of some work of H. König on that problem [*Math. Ann.* 128 (1955), 420-452; MR 16, 935], with applications to some of the distributions important for physics. It is essential for the following that the notion of product which results is in general non-unique, non-associative and non-commutative. The final part of the paper is a series of remarks on the application of the preceding ideas to field theory. The author at first considers the formal expressions which occur in Schwinger's equations for the propagators of electrodynamics [*Proc. Nat. Acad. Sci. U.S.A.* 37 (1951), 452-455, 455-459; MR 13, 520]. He proposes to interpret the propagation functions as distributions and the formal product as a distribution product. The non-uniqueness of the distribution product then results in the appearance of certain arbitrary constants which have the effect (in the special example considered) of a finite renormalization of mass. The author concludes that "it is in principle impossible to get uniquely determined values for the proper constants of a quantized system within the present relativistic field theory." In the final two paragraphs the author touches briefly on what to the reviewer is the main problem, that of giving a well defined mathematical meaning to the basic field operators of the theory. He expresses the opinion that the preceding definition of distribution product will make possible a treatment of products of singular operators in field theory. An Appendix is devoted to a summary of some results on the division problem for distributions.

A. S. Wightman (Princeton, N.J.).

Lehmann, H. Über Eigenschaften von Ausbreitungsfunktionen und Renormierungskonstanten quantisierter Felder. *Nuovo Cimento* (9) 11 (1954), 342-357.

This paper discusses some properties of propagators in local and non-local field theories with interaction satisfying the following assumptions: 1) the theory is Lorentz invariant; 2) the total energy operator possesses a minimum eigenvalue corresponding to the vacuum; 3) the theory is invariant under particle-anti-particle conjugation. No use is made of power-series expansions, and this, of course, is the important advance over most previous papers.

It is shown that the propagators cannot be less singular on the light cone than are the corresponding propagators for free fields, thus eliminating any hope of obtaining convergent results from non-renormalizable theories. The author shows that the Feynman propagators can be extended analytically into the lower half-plane, thus dispelling the doubt raised by Feldman [*Proc. Roy. Soc. London. Ser. A.* 223 (1954), 112-129] as to the possible occurrence of unrenormalizable divergences of the type which occur in certain approximate approaches to propagators. Formulas are given for the renormalization constants in terms of a function which is the basic new idea of the paper and may be described as an amplitude density for all states with the same absolute value of the energy-momentum vector. From these formulas new inequalities for the renormalization constants are obtained.

Finally, the author suggests a new method of successive approximations to the propagators and renormalization constants which is applied to obtain results to order g^2 for pseudo-scalar coupling of nucleons and mesons.

A. J. Coleman (Toronto, Ont.).

Nambu, Yoichiro. Structure of the scattering matrix.

Phys. Rev. (2) 98 (1955), 803-811.

This paper extends the results previously obtained by Källén [Helv. Phys. Acta 25 (1952), 417-434; MR 14, 435], Lehmann [in the paper reviewed above], and Gell-Mann and Low [Phys. Rev. (2) 95, 1300-1312 (1954); MR 16, 315]. These authors obtained parametric forms for vacuum expectation values of bilinear products of Heisenberg operators. In the present paper, similar parametric forms are obtained for matrix elements where the vacuum state is replaced by the one-particle state. The main tool used, in addition to the invariance properties of field theory, is the concept of microscopic causality, interpreted in the well specified sense, that two measurements at spatially separated points do not interfere. As an example of the type of results obtained, consider $M(K)$ defined as

$$M(K) \delta^4(k+p-l-q) = \frac{1}{(2\pi)^4} \iint e^{-i12} e^{ikx'} (q|T\phi(x)\phi(x')|p).$$

The momenta p and q specify the one-particle states, $K = \frac{1}{2}(k+1)$ and T is the chronological ordering operator. It is shown that $M(K)$ can be represented parametrically as

$$M(K) = -i \iiint \frac{e_1 + e_2(K \cdot P) + e_3(K \cdot Q)}{(K + \alpha P + \beta Q)^2 + m^2 - i\epsilon} d^3m^2 dx d\beta.$$

Here $P = \frac{1}{2}(k+q)$, $Q = \frac{1}{2}(k-q)$; e_1, e_2, e_3 are functions of the variables m^2, α and β ; e_1 and e_2 are real, and the ranges of α and β are restricted by $|\alpha| + |\beta| Q^2 / (\mu^2 + Q^2) \leq 1$. The functional form of the e 's is, however, unspecified. In fact, they may even contain derivatives of the delta function.

A. Salam (Cambridge, England).

Sunakawa, Sigenobu, Imamura, Tsutomu, and Utiyama, Ryōyū. Renormalization of two-electron Green-function.

Progr. Theoret. Phys. 12 (1954), 642-652.

The same authors in a previous paper [same journal 8 (1952), 77-110; MR 15, 82] proposed a new method for grouping together the renormalization terms in perturbation expansions of quantum field theory. The method is here applied to the two-particle Green-function, which is proved to be finite after renormalization. F. J. Dyson.

Fradkin, E. S. On the asymptotics of Green functions in quantum electrodynamics.

Z. Eksper. Teoret. Fiz. 28 (1955), 750-752. (Russian)

This is a brief summary of some results on the asymptotic behavior of Green's functions in quantum electrodynamics, simplifying the analysis of L. D. Landau, A. A. Abrikosov and I. M. Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) 95 (1954), 773-776; 96 (1954), 261-264; MR 16, 316]. The question, whether these results imply that the renormalized electron charge is zero, is also discussed briefly; the author considers this conclusion probable but not certain. The following important suggestion is made in passing. Suppose it to be true that conventional quantum electrodynamics rigorously implies $e=0$. Then we may expect to find, in a modified theory which avoids the divergence difficulties, a natural explanation of the fact that $e^2/\hbar c = 1/137$ is almost zero, i.e. small compared with unity.

F. J. Dyson (Princeton, N.J.).

Freese, E. Many-point correlation-functions in quantum field theory.

Nuovo Cimento (10) 2 (1955), 50-57.

This paper describes a method of dealing with scattering problems involving particles of low energies or large

angular momenta. The method is based on a new approximation procedure, which is justified by the author by using some vague physical arguments. The author believes that his treatment can be expected to give better results than the usual perturbational treatment in the case of intermediate couplings such as the meson-nucleon coupling.

S. N. Gupta (Lafayette, Ind.).

Loinger, A. Un'analogia fra l'elettrodinamica quantistica e l'elettrodinamica classica della descrizione corpuscolare.

Nuovo Cimento (10) 2 (1955), 511-518.

Morpurgo has shown [Nuovo Cimento (9) 9 (1952), 808-817; MR 14, 520] that in the dipole approximation the expectation value of the position operator of a point non-relativistic electron satisfies the Dirac-Lorentz equation. Here, this result is shown to hold independently of the dipole approximation. The author concludes that the well known "run away" solutions of the Dirac-Lorentz equations will also occur in quantum theory. (In the opinion of the reviewer the proof of this conclusion is incomplete since the author does not show that any state of the quantum-mechanical system exists in which the expectation value of the position operator "runs away".) A possible method of generalization of the results of the paper to the relativistic case is discussed briefly and rejected.

A. S. Wightman (Princeton, N.J.).

Lozano, J. M., and Medina N., F. M. Boundary conditions and S-formalism in nuclear scattering.

Rev. Mexicana Fis. 1 (1952), 102-113. (Spanish)

Central scattering with zero angular momentum of a scalar particle can be described by a function $S(k)$. The author guesses a function depending on time which is expressed in terms of $S(k)$ and verifies, in the case of one-resonance-level scattering, that it is identical with the time-dependent wave-function as given by Moshinsky [Phys. Rev. (2) 84 (1951), 525-533; MR 13, 610]. He then shows that as long as $S(k)$ is expressible in terms of Wigner's function $R(k^2)$, that is even if it has an essential singularity at infinity, the above time-dependent function has the correct initial and final values for the scattering process described by $S(k)$.

A. J. Coleman.

★ Moshinsky, Marcos. On a dynamical theory of scattering.

New research techniques in physics, pp. 285-298. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para América Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15-29, 1952. Rio de Janeiro, 1954.

The wave function $\psi(r, k, t)$ for a non-steady scattering process is expanded in the form

$$\psi(r, k, t) = \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos \theta) \psi_n(r, k, t).$$

Expressions are given for the ψ_n which generalize the formula of Lozano and Medina [in the paper reviewed above] for ψ_0 and reduce to known results for steady scattering. The explicit formulas for ψ_n , in the relativistic and nonrelativistic cases, are expressed in terms of functions occurring in the principal parts associated with the poles of the scattering matrix.

A. J. Coleman.

Phillips, R. J. N. Relativistic treatment of the vector meson field.

Nuovo Cimento (9) 12 (1954), 905-914.

The author applies the formalism developed by Dirac

[Phys. Rev. (2) 73 (1948), 1092-1103; Canad. J. Math. 2 (1950), 129-148; MR 10, 225; 13, 306] to the quantization of the vector meson field equations. The theory developed is compared to that of Stückelberg [Helv. Phys. Acta 11 (1938), 299-328]. It is shown that the latter theory cannot be generalized to a manifestly covariant form but under certain restrictions gives the same results as the theory of the author. *A. H. Taub* (Urbana, Ill.).

Glaser, Walter. Licht und Materie in einheitlicher Deutung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. 163 (1954), 215-265.

It is shown by means of detailed calculations that a light quantum which is trapped in a finite volume behaves like a material particle of finite rest-mass as regards its energy, mass and momentum. If a region containing a standing wave is viewed by an observer in relative motion, he will find the wave described as a wave group in the form of a modulated de Broglie wave, with a group velocity equal to the velocity of the region and with the energy and momentum of the group given according to relativistic mechanics. The author proposes a tentative particle model based on these results. He considers solutions of the Maxwell equations inside a small sphere for the case of zero energy-flux through the surface and obtains a spectrum of possible particle radii. The electromagnetic angular momentum is identified with the spin of the particle. Finally, the Maxwell equations are generalized so as to lead to the Schrödinger-Gordon equation and then to the Dirac equation for the electron.

N. Rosen (Haifa).

Wick, G. C. Introduction to some recent work in meson theory. Rev. Mod. Phys. 27 (1955), 339-362. Expository paper.

Falk, Gottfried. Zur Quantenmechanik des Mehrkörperproblems. Z. Physik 142 (1955), 297-309.

As is well known, the vectors in Hilbert space which are usually used to describe states of a quantum-mechanical system can also be regarded as complex-valued functions on the operators representing observables of the system. The purpose of the paper under review is twofold: to illustrate how the second point of view works in practice and to derive five theorems about the non-relativistic many-body problem. A typical one of these is the following: Consider a system of two similar compound particles each made of n fermions. Suppose the hamiltonian of the system is symmetric under exchange of the centers of gravities of the compound particles. Then under that exchange the wave function of the system is: (1) symmetric (anti-symmetric) if the two compound particles possess the same internal states in the limit of no interaction and n is even (odd); (2) symmetric or anti-symmetric, the two possibilities occurring in equal numbers, if in the limit of no interaction the two compound particles are in different states.

A. S. Wightman (Princeton, N.J.).

Królikowski, W., and Rzewuski, J. Covariant one-time formulation of the many-body problem in quantum theory. Nuovo Cimento (10) 2 (1955), 203-219.

The covariant four-dimensional partial integro-differential equation of the many-body problem in quantum field theory is shown to be equivalent to a covariant integro-differential equation containing surface integrals over one space-like surface only. Thus one can convert the many-time theory into a one-time scheme. The transfor-

mation holds for all values of the coupling constant except for certain characteristic ones. The kernel of the one-time equation is given in terms of a power expansion of the coupling constant and its unique relationship to the many-time kernel is discussed. The stationary treatment of the one-time equation is also discussed in view of previous approaches to this problem.

M. J. Moravcsik (Ithaca, N.Y.).

Robinson, R. O. A. The Bethe-Salpeter equation for many-body systems. Canad. J. Phys. 33 (1955), 369-382.

The procedure given by Wentzel [Phys. Rev. (2) 89 (1953), 684-688] for the reduction of a three-body Bethe-Salpeter equation to the Schrödinger form is extended to include relativistic corrections. A formal technique is developed, appropriate to the study of the corrections of the ground states of light atoms, for distinguishing between those terms which are included as iterations of the kernel and those which have to be inserted explicitly as perturbations. *P. T. Matthews*.

Nishijima, Kazuhiko. Many-body problem in quantum field theory. III. Progr. Theoret. Phys. 13 (1955), 305-328.

The method used previously by the author [same journal 10 (1953), 549-574; 12 (1954), 279-310; MR 15, 589; 16, 548] for treating dressed particles in quantum field theory is applied to the interaction of a scalar meson field with a scalar photon field. When supplemented by the device of introducing an external c -number field as a mathematical convenience, the method is shown to lead to the normalization of Wick's solution [Phys. Rev. (2) 96 (1954), 1124-1134; MR 16, 655] of the Bethe-Salpeter equation. Cross sections are computed for disintegration by a scalar photon of a system consisting of two scalar mesons and for the elastic scattering of such by a Coulomb field. An expression is obtained without ultraviolet divergences for the self-energy of such a composite system. The definitions of "in" and "out" used previously by the author are modified to give the correct asymptotic forms for the Feynman amplitudes.

H. C. Corben (Pittsburgh, Pa.).

Fok, V. A. On the Schrödinger equation for the helium atom. Izv. Akad. Nauk SSSR. Ser. Fiz. 18 (1954), 161-172. (Russian)

The Hylleraas variational method gives the lowest energy level of He to within experimental error [Chandrasekhar and Herzberg, Phys. Rev. (2) 98 (1955), 1050-1054] even though it is known that a series of the form assumed by him cannot satisfy the wave equation [Bartlett, Gibbons, and Dunn, *ibid.* 47 (1935), 679-680; Bartlett, *ibid.* 98 (1955), 1067-1070]. Fok proves that there is a solution of the He wave equation of the form $\sum_n e^{n(\ln \varrho)} f_n(\alpha, \theta)$, where $n \geq 0$, $0 \leq \alpha \leq [n/2]$, $\varrho = r_1^2 + r_2^2$, $\alpha = 2 \tan^{-1}(r_2/r_1)$ and θ is the angle between the vectors r_1 and r_2 from the nucleus to the two electrons. An infinite set of equations is obtained for f_n which can be solved successively in terms of four-dimensional spherical harmonics. The author makes no reference to the papers of Gronwall [*ibid.* 51 (1937), 655-660] and Bartlett [*ibid.* 51 (1937), 661-669] in which the existence of a series solution of the above form is proved by a more complicated method. Fok's paper concludes with a suggested ansatz for the variational approach which might be

expected to provide a more satisfactory approximation to the wave function than does that of Hylleraas.

A. J. Coleman (Toronto, Ont.).

Herpin, André, et Mercier, Claude. Quelques résultats relatifs à un opérateur non hermitique défini sous forme implicite. C. R. Acad. Sci. Paris 241 (1955), 177-178.

A special non-hermitian operator defined by an implicit operator equation and suggested by a nuclear model is studied and properties of its eigenfunctions are given.

L. Van Hove (Utrecht).

Román, P. Erhaltungsgesetze und quantenmechanische Operatoren. Acta Phys. Acad. Sci. Hungar. 5 (1955), 143-158. (Russian summary)

When the equations of motion are derivable from a variational principle (Hamilton's principle), it is also possible to derive the conservation laws from that principle [E. Noether, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1918, 235-257; summarized by E. L. Hill, Rev. Mod. Phys. 23 (1951), 253-260; MR 13, 503]. The author shows that this derivation can be carried through not only in quantum field theory, where this method is well known, but also in quantum mechanics. But whereas in quantum field theory one is thereby lead to the definition of the associated operators, in quantum mechanics one obtains the expectation values from which the operators can then be inferred. The method is also applicable to systems with Lagrangians involving higher derivatives than the first, and can be generalized to systems interacting with external sources or fields.

F. Rohrlich (Iowa City, Ia.).

Nagy, K. Die Quantentheorie der elektromagnetischen Strahlung in Dielektrika. Acta Phys. Acad. Sci. Hungar. 5 (1955), 95-118. (Russian summary)

The electromagnetic field in a dielectric moving with a uniform velocity is quantized. Previously J. M. Jauch and K. M. Watson [Phys. Rev. (2) 74 (1948), 950-957; MR 10, 346], in a similar work, made use of the canonical energy-momentum tensor of the field and obtained the result that the energy of a photon can be negative under certain conditions. In the present work, first the Abraham energy-momentum tensor is used, leading to the diagonalization of the energy, but not the momentum; then the "radiation" energy-momentum tensor is introduced instead, and both energy and momentum are diagonalized. It is found that in the general case a photon has a finite, positive, rest-mass.

N. Rosen (Haifa).

Bellomo, E. Sul moto di un elettrone finito e la corrispondenza con l'elettrone puntiforme nella meccanica classica relativistica. Nuovo Cimento (10) 2 (1955), 456-466.

The author studies a classical model of an extended electron invariant under special relativity with a view to understanding the origin of the "runaway solutions" of the Lorentz-Dirac equation of motion of a point electron. He derives an approximate equation of motion which includes the effect of the electron's self-interaction. From this equation it follows that the "runaway solutions" occur when $0 < m_0 + 4/3m_{e.m.} < 4/3m_{e.m.}$, where m_0 is an effective mechanical mass of the electron and $m_{e.m.}$ is the electromagnetic mass. Instructive pictures are given which show how the balance between the positive energy four-vector of the electromagnetic momentum and the

negative energy four-vector of the mechanical momentum is unstable in the runaway case.

A. S. Wightman.

Allcock, G. R., and Kuper, C. G. 'Rotons' in quantum hydrodynamics. Proc. Roy. Soc. London. Ser. A. 231 (1955), 226-243.

The authors diagonalize the quantum-mechanical Hamiltonian obtained by Ziman [same Proc. 219 (1953), 257-270] and Thellung [Physica 19 (1953), 217-226] for an ideal liquid in the approximation of small compressibility. Expansions in powers of s^{-1} are used where s is the sound velocity. The roton rest energy is found to be proportional to the inverse of the sound velocity and to the ninth power of the wave number cut-off.

A. H. Taub (Urbana, Ill.).

See also: Waugh, p. 276; Rose, p. 326.

Thermodynamics, Statistical Mechanics

★ Brønsted, J. N. Principles and problems in energetics. Translated from the Danish by R. P. Bell. Interscience Publishers, New York-London, 1955. vii+119 pp. \$3.50.

This book strongly criticizes the current logical foundations of thermodynamics, in particular, the treatment by Planck, Carathéodory, and Born (the work of the Onsager school is not mentioned). It advocates the use of potentials P for quantities K transported so that $\sum (P_1 - P_2) \delta K = 0$ for every reversible process in a closed system, δK being the quantity transported, and P_1, P_2 the values of the corresponding potential in the initial and final states. If the process is not reversible, the sum is set equal to $T \delta S$, the "energetic heat evolution", δS being the entropy production, and T the absolute temperature. The reviewer has not been able to find a noncircular definition of the latter in the book, nor a proof of the existence of entropy. The presentation in the book is not formalized, terms are used before they are defined, metatheoretical remarks are not separated from the theory itself, and there are baffling repetitions. The last four sections are entitled: The thermoelectric process, The thermoelectric cell, The electrochemical cells, Thermochemical cells, and seem to this reviewer to abound in unstated assumptions.

A. W. Wundheiler (Chicago, Ill.).

Wergeland, Harald. Fluctuations in the external reactions of a thermodynamical system. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 106-111.

A well-known formula of Gibbs [Collected works, vol. II, part I, Longmans-Green, London, 1928, pp. 81, 82] for computing the fluctuations of a thermodynamic force A is shown to give ambiguous results in many cases, particularly for the pressure fluctuations of an ideal gas in a box. The formula in question is

$$\overline{(A - \bar{A})^2} = \Theta \left(\frac{\partial^2 E}{\partial a^2} - \frac{\partial^2 \psi}{\partial a^2} \right)$$

in which $\Theta = kT$, a is any external parameter, A its conjugate force, χ the Gibbs free energy, E the energy, and average of phase functions are evaluated with a canonical distribution.

The author concludes that the partial derivatives of E should be evaluated while keeping only the set of all

action integrals of the system constant rather than while keeping the phase point of the system constant as prescribed by Gibbs. This revised scheme leads actually to simpler formulas,

$$(\overline{p-p})^2 = \Theta \left[\left(\frac{\partial \bar{p}}{\partial V} \right)_\Theta - \left(\frac{\partial \bar{p}}{\partial V} \right)_s \right]$$

for the pressure fluctuation in terms of the difference between the isothermal and isoentropic tension coefficients and in general

$$(\overline{A-A})^2 = -\Theta \left[\left(\frac{\partial \bar{A}}{\partial a} \right)_\Theta - \left(\frac{\partial \bar{A}}{\partial a} \right)_s \right]$$

in which $\bar{\eta}$ is the Gibbs index on the canonical distribution.

This results is quite significant for two reasons. First because the fluctuations are expressed directly in terms of thermodynamic observables and secondly because errors in use of the Gibbs formula appear in many text books including some very recent ones. *G. Newell.*

Dedebant, Georges. *Le principe de Carnot, du point de vue aléatoire.* C. R. Acad. Sci. Paris 241 (1955), 355-356.

A simplified stochastic model is adopted for a gas functioning as heat engine between two thermostats and the output is calculated. *L. Van Hove (Utrecht).*

Țițeica, Șerban. *Le troisième principe de la thermodynamique et la mécanique statistique.* Rev. Math. Phys. 2 (1954), 18-25 (1955).

Let W be the energy of a system, with $W=0$ for the lowest state. Let $P(w)$ be the number of states (extension in phase space) for which $W < w$. Then the sum of states $Z(\beta) = \int \exp(-\beta W) dP(W)$ determines the entropy according to $S(\beta) = k \log Z - k\beta(d \log Z/d\beta)$. The author shows that for $\beta \rightarrow +\infty$ one has $S \rightarrow -\infty$, unless $P(+0) - P(0) > 0$. The conclusion is that the so-called third law of thermodynamics can only be valid if the allowed energy values form a discrete spectrum, or if at least the lowest value has a finite a priori probability.

N. G. van Kampen (Utrecht).

Ecker, G. *Zur statistischen Beschreibung von Gesamtheiten mit kollektiver Wechselwirkung. I. Grundlagen und Grenzen kollektiver Beschreibung.* Z. Physik 140 (1955), 274-292.

The paper gives an extensive discussion of the various assumptions needed for the statistical treatment of collective states of motion in assemblies of particles. After a survey of the literature on collective motion of electrons, a critical analysis is made of the following points: the introduction of a particle distribution function in phase space, the validity of the Maxwell-Boltzmann transport equation, and the use of the average force on a particle as first proposed by Vlasov for plasma oscillations. The conditions of validity required for these various steps are written down. They are applied in the following paper to the special case of electron plasmas.

L. Van Hove (Utrecht).

Ecker, G. *Zur statistischen Beschreibung von Gesamtheiten mit kollektiver Wechselwirkung. II. Die Bedeutung der Beschränkungen des D-Modells für die Begriffsbildung und Ergebnisse kollektiver Beschreibung.* Z. Physik 140 (1955), 293-307.

In continuation of the paper reviewed above the

conditions there derived for the applicability of the statistical theory of collective motion are discussed for the special cases of plasma oscillations. Attention is paid to the discrepancies between the dispersion formulae found by various authors and to the divergencies occurring in some of them.

L. Van Hove (Utrecht).

Pompéia, Paulus A. *Distribution problems related to statistical physics.* An. Acad. Brasil. Ci. 27 (1955), 123-136.

Expressions are given for the number of ways in which N particles can be distributed among M cells, if the cells are indistinguishable (i.e., two distributions are counted the same if the numbers of cells with 0, 1, 2, ... particles are the same in both). As to the particles, both cases, distinguishable or not, are considered. Application to statistical mechanics gives the usual results. Experimental data concerning the statistical distribution of discharges in Geiger counters are compared with this theory. No reason is offered why the current statistics should have to be supplanted by the one here proposed.

N. G. van Kampen (Utrecht).

Olsen, Haakon. *Note on multiple scattering theories.*

Norske Vid. Selsk. Forh., Trondheim 28 (1955), 5-9.

The multiple scattering of particles in a flat sheet of material may be considered as a Markov process, if back scattering is neglected. Accordingly, the angular distribution function satisfies a Chapman-Kolmogorov equation, which is the small-angle approximation to what Chandrasekhar calls 'principles of invariance' [Radiative transfer, Oxford, 1950; MR 13, 136]. The existing theories of small-angle scattering are shown to follow from this general point of view. [It appears that the word 'obviated' in the last section is to be read as 'made obvious'.]

N. G. van Kampen (Utrecht).

Sáenz, A. W., and O'Rourke, R. C. *Number of states and the magnetic properties of an electron gas.* Rev. Mod. Phys. 27 (1955), 381-398.

The sum of states is the Laplace transform of the level density; conversely, the number of states up to the energy E is

$$N(E) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{zE}}{z} \text{trace } e^{z\mathcal{H}} dz$$

The authors employ this equation to compute $N(E)$ for an assembly of electrons with total Hamiltonian $\mathcal{H}_T = \mathcal{H}_0 + \mathcal{H}_s + \mathcal{H}_1$. The spin part \mathcal{H}_s gives no difficulties, as it commutes with \mathcal{H}_T . The interaction \mathcal{H}_1 is treated as perturbation (in the way the operator $\exp[-it(H_0 + H_1)/\hbar]$ is expanded in field theory; this expansion is attributed to Schwinger, but was previously used by Born [Z. Physik 40 (1926), 167-192]). A second method of expanding $\exp[-z(\mathcal{H}_0 + \mathcal{H}_1)]$ in \mathcal{H}_1 is also given.

The resulting formulas are applied, firstly to calculate $N(E)$ for an electron in a periodic potential, to second order in this potential. Secondly, $N(E)$ is calculated for a free-electron gas in a constant magnetic field, and used to obtain the familiar expression for its magnetic moment.

N. G. van Kampen (Utrecht).

See also: Pekeris, p. 273.

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